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Zusammenfassung

In der vorliegenden Arbeit werden eindimensionale Störungen von nichtnegativen Operatoren in Kreinräumen betrachtet. Dabei wird untersucht wie sich die Anzahl der Eigenwerte und deren Vielfachheit in einer Lücke des essentiellen Spektrums unter einer Störung ändern können. Zudem wird beschrieben wie sich an einem Eigenwert die Anzahl und die Länge der linear unabhängigen Jordanketten ändern können.

Für zwei selbstadjungierte Operatoren A und B in einem Kreinraum, sodass A nichtnegativ und die Resolventendifferenz von A und B eindimensional ist, werden die folgenden Ergebnisse erzielt:

1. Ist $I \subseteq \mathbb{R}$ ein offenes Intervall, so dass alle Punkte des Spektrums von A in I isolierte Eigenwerte und Pole der Resolvente von A sind, so verringert sich durch eine eindimensionale Störung die Anzahl der Eigenwerte von A in I um höchstens 2 und erhöht sich um höchstens 3. Diese Schranken werden besser, falls der gestörte Operator B ebenfalls nichtnegativ ist oder $0 \notin I$ gilt.
2. Die Summe der geometrischen Vielfachheiten der Eigenwerte von A in I verringert sich durch die Störung um höchstens 4 und erhöht sich um höchstens 6, wobei wiederum bessere Schranken erzielt werden, falls B nichtnegativ ist oder $0 \notin I$.
3. Die Jordanstruktur von A (das heißt die Anzahl und Länge der linear unabhängigen Jordanketten) zu einem Eigenwert $\mu \in I$ ändert sich durch die Störung nur auf 11 verschiedene Arten, wobei sich die Anzahl der möglichen Jordanstrukturen von B weiter verringert, falls $\mu \neq 0$ ist. Ist B zudem nichtnegativ, so gibt es nur 7 mögliche Jordanstrukturen, bzw. 3 falls $\mu \neq 0$ ist.

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Ilmenau, January 2016 Leslie Leben

Contents

Introduction

We study the spectrum of a non-negative operator A in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ under rank one perturbations in resolvent sense. The following two questions are answered:

(i) How does the spectral multiplicity in a gap of the essential spectrum of A change under rank one perturbations?

(ii) How does the Jordan structure at isolated eigenvalues of A change under rank one perturbations? More precisely, how does the number and the length of Jordan chains of A at a given eigenvalue change under a rank one perturbation?

Rank one and finite rank perturbations of selfadjoint operators in Hilbert spaces have been studied in many applications in theoretical physics, e.g. in the investigation of singular perturbations in quantum mechanics, see [2, 3, 4, 19, 32, 40, 49, 55, 56, 71, 94]. It is well known that an n-dimensional selfadjoint perturbation of a selfadjoint operator in a Hilbert space preserves the essential spectrum and changes the spectral multiplicity by at most n. More precisely, for a bounded interval $I \subseteq \mathbb{R}$ and selfadjoint operators A, B in a Hilbert space $\mathcal{H}$ such that

$$(A - \lambda_0)^{-1} - (B - \lambda_0)^{-1} \tag{1}$$

is of rank n for some $\lambda_0 \in \rho(A) \cap \rho(B)$, the dimensions of the spectral subspaces of A and B corresponding to the interval I differ at most by n, and this estimate is sharp. In particular, if $I \subseteq \rho(A)$ then I contains at most n isolated eigenvalues of B counted with multiplicities.

In the general non-selfadjoint case rank one and finite rank perturbations preserve the essential spectrum but precise results on the number and multiplicity of the discrete spectrum do not exist. Without further assumptions on the structure of the operators or the rank one perturbation the number of eigenvalues in a given interval can change arbitrarily, see [70, Theorem 1]. If the operators A and B under consideration are not selfadjoint in a Hilbert space but still selfadjoint in a Krein space, then several results on finite rank perturbations of different classes of operators exist, cf. [7, 8, 10, 11, 21, 31, 43, 62, 63, 64, 65]. However, these perturbation results are typically of qualitative nature and do not contain explicit bounds or estimates on the number and multiplicities of eigenvalues after the perturbation.

We consider the following situation, which is slightly more general than in Question (i).

Assumption (A). Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that A is non-negative and (1) is of rank one for some $\lambda_0 \in \rho(A) \cap \rho(B)$. Let $I \subseteq \mathbb{R}$ be an open interval and assume that $\rho(B) \cap I \neq \emptyset$ and $\sigma(A) \cap I$ consists only of isolated eigenvalues.

In this setting our answer to Question (i) is the following: The difference of the number $n_A(I)$ of distinct eigenvalues of A in I and the number $n_B(I)$ of distinct eigenvalues of B in I can be estimated by the number $n_{A,B}(I)$ of common eigenvalues of A and B in I and a correction term which is at most 3. The correction term depends on the fact whether 0 is in the interval I and whether the operator B is non-negative ($\kappa_B = 0$) or has one negative square ($\kappa_B = 1$):

(i) If $0 \notin I$ then

$$n_A(I) - n_{A,B}(I) - 1 \le n_B(I) \le n_A(I) + n_{A,B}(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(ii) If $0 \in I$ then

$$n_A(I) - n_{A,B}(I) - 2 \le n_B(I) \le n_A(I) + n_{A,B}(I) + \begin{cases} 2 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

It is remarkable that all the estimates above turn out to be sharp: There exist operators A and B (which are in fact matrices) such that the inequalities in (i) and (ii) become equalities. Moreover, we mention that the above estimates imply that the finiteness of the number of distinct eigenvalues of A in a gap of the essential spectrum is preserved under a rank one perturbation. This is a special case of a more general result from [21].

The second main result are estimates of the total algebraic multiplicities $m_A(I)$ and $m_B(I)$ of the eigenvalues of A and B in I. This leads to the following estimates in Section 5.3 on the multiplicities of the eigenvalues:

(i) If $0 \notin I$ then

$$m_A(I) - 1 \le m_B(I) \le m_A(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(ii) If $0 \in I$ and $0 \notin \sigma_p(A)$ then

$$m_A(I) - 2 \le m_B(I) \le m_A(I) + \begin{cases} 2 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(iii) If $0 \in I$ and $0 \in \sigma_p(A)$ then

$$m_A(I) - 4 \le m_B(I) \le m_A(I) + \begin{cases} 4 & \text{if } \kappa_B = 0, \\ 6 & \text{if } \kappa_B = 1. \end{cases}$$

For the proof of the above bounds we show that the dimensions of the root subspaces $\mathfrak{L}_0(A)$ and $\mathfrak{L}_0(B)$ of A and B at 0 differ at most by two,

$$\left|\dim \mathfrak{L}_0(A) - \dim \mathfrak{L}_0(B)\right| \le 2, \tag{2}$$

which is somehow a surprise as A and B may have Jordan chains at 0 of length up to 2 and 4, respectively.

The inequality (2) itself is a consequence of a very detailed analysis of $\mathfrak{L}_0(A)$ under a rank one perturbation, which leads to Question (ii). For its answer we show in a first step a bound on the change of the number of linearly independent Jordan chains: Consider the space $\ker(A-\lambda)^{p+1}/\ker(A-\lambda)^p$. Its dimension coincides with the number of linearly independent Jordan chains of A at λ of length at least $p+1$. We show in Theorem 3.1 that the change of the number of these Jordan chains of A at λ under a rank one perturbation is bounded by 1,

$$\left|\dim\left(\frac{\ker(A-\lambda)^{p+1}}{\ker(A-\lambda)^p}\right)-\dim\left(\frac{\ker(B-\lambda)^{p+1}}{\ker(B-\lambda)^p}\right)\right|\le 1. \tag{3}$$

Relation (3) plays an important role in the answer of Question (ii). We mention that for finite-dimensional $\mathcal{K}$ (i.e. A and B are matrices) this was shown in [91, Lemma 2]. Moreover, there exists a lower bound for the dimension of the root subspace of the perturbed operator B in terms of the dimension of the root subspace of A and the length of the Jordan chains of A at λ, cf. [45, 91]. Such a result was also proven in [57, Theorem 3] in the case of a compact operator A.

The answer to Question (ii) is now the following: Let A, B, and I be as in Assumption (A) and let $\mu\neq 0$ be an isolated eigenvalue of A with finite algebraic multiplicity. Then by (3), the dimension of $\ker(B-\mu)$ is either one less, equal to, or one greater than the dimension of $\ker(A-\mu)$. Moreover, in Section 4.2 we show for the Jordan structure of B at μ:

(i) If $\kappa_B=0$ then B has no Jordan chain of length ≥ 2 at μ. Hence, there are 3 possible Jordan structures of B at μ.

(ii) If $\kappa_B=1$ and if the dimension of $\ker(B-\mu)$ is greater than the dimension of $\ker(A-\mu)$, then B may have a Jordan chain of length 2 or 3 at μ. In the other cases, B has no Jordan chain of length ≥ 2 at μ. Thus, there are 5 possible Jordan structures of B at μ.

An analogous result holds for $\mu=0$, but due to the fact that A may have Jordan chains at 0, more Jordan structures of B are possible: As above, the dimension of $\ker B$ is either one less, equal to, or one greater than the dimension of $\ker A$, cf. (3). Analogously, at 0 B either has one less Jordan chain of length 2, equally many Jordan chains of length 2, or one more Jordan chain of length 2 than A. Moreover, we show in Section 4.1:

(i) If $\kappa_B=0$ then all Jordan chains of B at 0 are of length at most 2 and there are only 7 possible Jordan structures of B at 0.

(ii) If $\kappa_B=1$ then B has at most one Jordan chain at 0 of length 3 or 4. If such a Jordan chain exists then $\ker B$ is contained in $\ker A$ and $\dim\mathfrak{L}_0(A)=\dim\ker A^2\ge\dim\ker B^2$. Moreover, A does not have more linearly independent Jordan chains of length 2 at 0 than B and if A and B have equally many linearly independent Jordan chains of length 2 at 0, then $\ker B$ and $\ker A$ coincide. In total, there are 11 possible Jordan structures of B at 0.

For the precise descriptions of the possible Jordan structures see Theorem 4.7.

For matrices (and compact operators) there are various results on the structure of root subspaces under perturbations. In general, the multiplicity of a given eigenvalue can change arbitrarily, see [70, Theorem 2], but in [57] Hörmander and Melin showed for compact operators A that so-called *generic* rank k perturbations destroy the k longest Jordan chains of A. Possibly remaining Jordan chains remain intact and new eigenvalues are simple. Here, a set $\Omega \neq \mathbb{C}^n$ is called generic if its complement is contained in an algebraic set. In the matrix case, this result was independently reproved by Dopico and Moro in [45] and Savchenko [89, 90]. Today generic perturbation theory for matrices has become a topic of high interest, see for example [44, 88]. In [81, 82, 83, 84, 85] the authors studied (real and complex) matrices that are structured (for example selfadjoint, skew-adjoint or unitary) with respect to an indefinite inner product in $\mathbb{C}^n$. Both structured and unstructured generic rank one perturbations of such matrices were investigated and the generic behaviour of root subspaces under perturbation described. In [47] results on non-generic perturbations of matrices are shown. The authors investigate "nearby" Jordan structures, i.e. Jordan structures which can occur by a small perturbation of a given Jordan structure, and use these theoretical results to enhance the numerical computation of Jordan normal forms of matrices.

The thesis is organised as follows. In Chapter 1 we give a short introduction to the theory of Krein spaces and operators therein. In Section 2.1 we introduce the concepts of linear relations and boundary triplets in Krein spaces. For a boundary triplet exists a so-called Weyl function which we use in Section 2.2 to express the resolvent difference of two selfadjoint operators A and B in a Krein space which differ by a rank one operator. Here the Weyl function M_A is scalar. Roughly speaking the poles (zeros) of M_A coincide with the isolated eigenvalues of A (B, respectively). In Section 2.3 we explore the connections between the sign types of isolated spectral points of A and B, and the behaviour of the function M_A at its poles and zeros. In Section 2.4 we show roughly speaking, how for a given rational scalar function g, a boundary triplet can be constructed such that the associated Weyl function equals g. Chapter 3 is devoted to the study of root subspaces under finite rank perturbations. In particular we show (3) in Section 3.1. We continue the investigation in the case of rank one perturbations in Section 3.2. These results are then applied in Chapter 4 to a non-negative operator A under a rank one perturbation. In Section 4.1 we present a detailed analysis of the Jordan structure of the perturbed operator B at 0, which is one of the main results of this thesis. An analogous result for root subspaces at real non-zero points is given in Section 4.2. After some preparations in Section 5.1 we show Theorems 5.12 and 5.18 in Section 5.2 and 5.3. Finally, in Chapter 6 we combine the results on root subspaces and spectral intervals under rank one perturbations.

Section 3.1 is contained in a joint work with Behrndt, Martínez Pería, and Trunk, see [18]. Sections 2.3, 5.2, and 5.3 of this thesis are contained in a joint work with Behrndt, Martínez Pería, Möws, and Trunk, see [17].

1. Krein Spaces

In this chapter we introduce the concepts of Krein spaces and operators therein. For a detailed exposition see [5, 9, 26, 58, 59, 73]. We will concentrate on different subclasses of selfadjoint operators and present some of their spectral properties.

1.1 The Geometry of Krein Spaces

The structure of a Krein space is given by an indefinite inner product. The resulting geometric properties are investigated in this section.

Definition 1.1. Let $\mathcal{K}$ be a vector space. A map $[\cdot,\cdot] : \mathcal{K} \times \mathcal{K} \to \mathbb{C}$ satisfying

(i) $[\lambda x + \mu y, z] = \lambda[x,z] + \mu[y,z]$ for all $x, y, z \in \mathcal{K}$ and $\lambda, \mu \in \mathbb{C}$, and

(ii) $[x,y] = \overline{[y,x]}$ for all $x, y \in \mathcal{K}$,

is called *inner product* on $\mathcal{K}$. The inner product $[\cdot,\cdot]$ is *non-degenerated*, if $[x,y] = 0$ for all $y \in \mathcal{K}$ implies $x = 0$.

Let $\mathcal{K}$ be a vector space with inner product $[\cdot,\cdot]$. We call a vector $x \in \mathcal{K}$ *positive* (*negative*, *neutral*), if $[x,x] > 0$ ($[x,x] < 0$, $[x,x] = 0$, respectively). A subspace $\mathcal{L} \subseteq \mathcal{K}$ is called *non-negative* (*non-positive, neutral*) if $[x,x] \geq 0$ ($[x,x] \leq 0$, $[x,x] = 0$, respectively) for every $x \in \mathcal{L}$. Moreover, $\mathcal{L}$ is called *positive* (*negative*), if $[x,x] > 0$ ($[x,x] < 0$, respectively) for every $x \in \mathcal{L}\backslash\{0\}$. For an arbitrary set $M \subseteq \mathcal{K}$, the *orthogonal complement of M with respect to* $[\cdot,\cdot]$ is

$$M^{[\perp]} := \{x \in \mathcal{K} \,|\, [x,y] = 0 \text{ for all } y \in M\}.$$

Note, that $M^{[\perp]}$ is a subspace. We say the elements $x, y \in \mathcal{K}$ are *orthogonal with respect to* $[\cdot,\cdot]$, (shortly $x[\perp]y$), if $[x,y] = 0$. Two subspaces $\mathcal{L}, \mathcal{M} \subseteq \mathcal{K}$ are called *orthogonal with respect to* $[\cdot,\cdot]$ or $\mathcal{L}[\perp]\mathcal{M}$, if $x[\perp]y$ for all $x \in \mathcal{L}$ and all $y \in \mathcal{M}$. If in addition $\mathcal{L} \cap \mathcal{M} = \{0\}$ holds, we denote the *direct orthogonal sum* of the subspaces by $\mathcal{L}[\dot{+}]\mathcal{M}$. The space $\mathcal{L}^\circ := \mathcal{L} \cap \mathcal{L}^{[\perp]}$ is the *isotropic part* of $\mathcal{L}$.

Definition 1.2. Let $\mathcal{K}$ be a vector space over $\mathbb{C}$ and $[\cdot,\cdot]$ an inner product on $\mathcal{K}$. The pair $(\mathcal{K}, [\cdot,\cdot])$ is called *Krein space*, if there exist two subspaces $\mathcal{K}_+$ and $\mathcal{K}_-$, such that

$$\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$$

and $(\mathcal{K}_+, [\cdot,\cdot])$, $(\mathcal{K}_-, -[\cdot,\cdot])$ are Hilbert spaces. Such a decomposition is called *fundamental decomposition* of $\mathcal{K}$. If $\dim \mathcal{K}_+ < \infty$ or $\dim \mathcal{K}_- < \infty$ holds, we call $\mathcal{K}$ a *Pontryagin space.*

Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space and $\mathcal{K} = \mathcal{K}_+[\dotplus]\mathcal{K}_-$ be a fundamental decomposition of $\mathcal{K}$. Such a decomposition is not unique. But for every other fundamental decomposition $\mathcal{K} = \mathcal{K}'_+[\dotplus]\mathcal{K}'_-$ we have $\dim \mathcal{K}_\pm = \dim \mathcal{K}'_\pm$ (cf. [73, §I.1]). We write each $x \in \mathcal{K}$ as $x = x_+ + x_-$ with $x_\pm \in \mathcal{K}_\pm$ and define

$$(\cdot,\cdot) : \mathcal{K} \times \mathcal{K} \to \mathbb{C}, \quad (x, y) := [x_+, y_+] - [x_-, y_-].$$

As $(\mathcal{K}_+, [\cdot,\cdot])$ and $(\mathcal{K}_-, -[\cdot,\cdot])$ are Hilbert spaces, $\mathcal{K}$ equipped with $(\cdot,\cdot)$ is also a Hilbert space. Moreover, it is easy to see, that the subspaces $\mathcal{K}_+$ and $\mathcal{K}_-$ are orthogonal with respect to $(\cdot,\cdot)$.

Let $P_\pm$ be the orthogonal (with respect to $(\cdot,\cdot)$) projections onto $\mathcal{K}_\pm$. The operator

$$J : \mathcal{K} \to \mathcal{K}, \quad J := P_+ - P_-,$$

is called *fundamental symmetry.* One easily calculates $J = J^{-1} = J^*$, where J^* denotes the adjoint of J with respect to $(\cdot,\cdot)$. Moreover, for all $x, y \in \mathcal{K}$ we have

$$[Jx, y] = (x, y), \quad [x, y] = (Jx, y), \quad \text{and} \quad [Jx, y] = [x, Jy].$$

For different fundamental decompositions of $\mathcal{K}$ we obtain different scalar products on $\mathcal{K}$ and corresponding induced norms, to which the inner product $[\cdot,\cdot]$ is continuous. By [73, Proposition I.1.2] all these induced norms are equivalent and hence induce the same topology on $\mathcal{K}$. We fix this topology and refer to it all upcoming topological terms. The induced norm to our chosen fundamental decomposition is denoted by $\|\cdot\|$.

Closed subspaces of Krein spaces are not necessarily Krein spaces again. But each closed subspace of a Krein space decomposes into a sum of its isotropic part, a positive and a negative subspace, see [9, Theorems I.6.4 and I.6.7].

Proposition 1.3. *Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space and $\mathcal{L} \subseteq \mathcal{K}$ a closed subspace. Then $\mathcal{L}$ admits a decomposition*

$$\mathcal{L} = \mathcal{L}^\circ[\dotplus]\mathcal{L}_+[\dotplus]\mathcal{L}_- \tag{1.1}$$

into the isotropic part $\mathcal{L}^\circ$, a positive subspace $\mathcal{L}_+$ and a negative subspace $\mathcal{L}_-$. Moreover, for any other decomposition $\mathcal{L} = \mathcal{L}^\circ[\dotplus]\mathcal{M}_+[\dotplus]\mathcal{M}_-$ we have $\dim \mathcal{L}_\pm = \dim \mathcal{M}_\pm$.

For the next proposition see [73, Section I.1, Corollary 1 and 2].

Proposition 1.4. *Let $(\mathcal{K} = \mathcal{K}_+[\dotplus]\mathcal{K}_-, [\cdot,\cdot])$ be a Krein space. Then we have for every non-negative (non-positive) closed subspace $\mathcal{L}_+ \subseteq \mathcal{K}$ ($\mathcal{L}_- \subseteq \mathcal{K}$)*

$$\dim \mathcal{L}_+ \leq \dim \mathcal{K}_+ \quad (\dim \mathcal{L}_- \leq \dim \mathcal{K}_-, \ \textit{respectively}).$$

1.2 Operators in Krein Spaces

The adjoint of a linear operator in a Krein space is defined in a similar way as in a Hilbert space. We recall the definition (see for example [73, §I.3] or [9, Definition 2.1.1]).

Definition 1.5. Let T be a closed, densely defined operator in the Krein space $(\mathcal{K}, [\cdot,\cdot])$. The Krein space *adjoint* T^+ of T is defined by

$$\begin{aligned}\operatorname{dom} T^+ &:= \{x \in \mathcal{K} \mid \text{there exists } x' \in \mathcal{K} \text{ such that for all } y \in \operatorname{dom} T \text{ we have } [Ty, x] = [y, x']\},\\ T^+x &:= x',\end{aligned}$$

where $\operatorname{dom} T$ denotes the *domain* of T.

As in a Hilbert space, one can show that the operator T^+ is linear and closed. We call T *symmetric* (*selfadjoint*) with respect to $[\cdot,\cdot]$, if $T \subseteq T^+$ ($T = T^+$, respectively). Often we will just say T is symmetric (selfadjoint) if it is clear which inner product is meant.

Remark 1.6. The Krein space adjoint does not depend on the fundamental symmetry J. However, we can derive a relation between T and T^+ relying on J: For $x \in \operatorname{dom} T^+$ exists by definition an $x' \in \mathcal{K}$, such that $(Ty, Jx) = [Ty, x] = [y, x'] = (y, Jx')$ for all $y \in \operatorname{dom} T$. Therefore, $Jx \in \operatorname{dom} T^*$ and $T^*Jx = Jx' = JT^+x$. Hence, $T^+ \subseteq JT^*J$. Since the reverse inclusion follows analogously, we have

$$T^+ = JT^*J.$$

For a closed operator T in $\mathcal{K}$ let $\sigma(T)$ and $\rho(T)$ denote the *spectrum* and *resolvent set* of T. By $E(\sigma, T)$ we denote the *Riesz-Dunford projection* corresponding to a spectral set σ and T, see [46, Section VII.9] (cf. also [50, Theorem XV.2.1] and [66, Theorem III.6.17]). For a set $\Omega \subseteq \mathbb{C}$ let $\Omega^* := \{\overline{z} \mid z \in \Omega\}$ be the complex conjugate set of Ω. The next proposition shows, that the spectrum of a selfadjoint operator in a Krein space has a certain symmetry, see [73, Proposition I.3.2].

Proposition 1.7. *Let T be a selfadjoint operator in $(\mathcal{K}, [\cdot,\cdot])$. Then $\sigma(T)$ is symmetric with respect to the real axis: $\sigma(T) = \sigma(T)^*$. If σ is a spectral set of T such that $\sigma = \sigma^*$ ($\sigma \cap \sigma^* = \emptyset$, respectively), then*

$$E(\sigma, T) = E(\sigma, T)^+ \quad (E(\sigma, T)^+E(\sigma, T) = 0, \ \textit{respectively}).$$

For a closed operator T let $\ker T$ be the *kernel* and $\operatorname{ran} T$ the *range* of T. For λ in the *point spectrum* $\sigma_{\mathrm{p}}(T)$ of T, denote by

$$\mathcal{L}_\lambda(T) := \bigcup_{n=1}^{\infty} \ker(T - \lambda)^n$$

the *root subspace* of T at λ. The *geometric* and *algebraic multiplicity* of λ are defined as usual as $\dim \ker(T - \lambda)$ and $\dim \mathcal{L}_\lambda(T)$, respectively. A *Jordan chain of T of length n at $\lambda \in \sigma_{\mathrm{p}}(T)$* is a finite ordered set of non-zero vectors $\{x_0, \dots, x_{n-1}\}$ contained in the root subspace $\mathcal{L}_\lambda(T)$ such that $(T-\lambda)x_0 = 0$ and $(T-\lambda)x_i = x_{i-1}$, $i = 1, \ldots, n-1$. The elements of a Jordan chain are linearly independent. The first $n-1$ elements of a Jordan chain of length n form a Jordan chain of length $n-1$. Furthermore, we say that T has k Jordan chains of length n at λ if there exist k linearly independent Jordan chains of length n at λ.

In general, selfadjoint operators in Krein spaces can have non-trivial Jordan chains. The following lemma shows, that the "upper half" of such a Jordan chain consists of neutral vectors.

Lemma 1.8. *Let T be a selfadjoint operator in $(\mathcal{K}, [\cdot,\cdot])$. If $\{x_0, \ldots, x_{n-1}\}$, $n \geq 2$, is a Jordan chain of T at some real eigenvalue λ, then $x_0, \ldots, x_m$ are neutral, where $m = \frac{n}{2} - 1$ if n is even and $m = \frac{n-1}{2} - 1$ if n is odd.*

Proof. Let $\{x_0, \ldots, x_{n-1}\}$ for some $n \geq 2$ be a Jordan chain of T at λ and $k \leq m$. Then $n-k-1 \geq k+1$ and

$$[x_k, x_k] = [x_k, (T-\lambda)^{n-k-1} x_{n-1}] = [(T-\lambda)^{n-k-1} x_k, x_{n-1}] = [0, x_{n-1}] = 0.$$

Hence x_k is neutral. □

With the help of selfadjoint projections we give a characterisation for subspaces of Krein spaces which are Krein spaces again, see [73, Theorem I.5.2].

Proposition 1.9. *Let $\mathcal{L} \subseteq \mathcal{K}$ be a closed subspace. Then the following are equivalent.*

(i) *$(\mathcal{L}, [\cdot,\cdot])$ is a Krein space.*

(ii) *$(\mathcal{L}^{[\perp]}, [\cdot,\cdot])$ is a Krein space.*

(iii) *$\mathcal{K} = \mathcal{L}[\dot{+}]\mathcal{L}^{[\perp]}$.*

(iv) *$\mathcal{L} = E\mathcal{K}$, with a projection E which is selfadjoint with respect to $[\cdot,\cdot]$.*

Among the selfadjoint operators in Krein spaces, there is the important subclass of definitizable operators, cf. for example [73, §I.3].

Definition 1.10. A selfadjoint operator T in $(\mathcal{K}, [\cdot,\cdot])$ is called *definitizable* if $\rho(T) \neq \emptyset$ and there exists a real-valued polynomial $p \neq 0$ such that

$$[p(T)x, x] \geq 0 \quad \text{for all } x \in \operatorname{dom}(T^k),$$

where $k := \deg p$. Such a polynomial is called *definitizing* for T.

A definitizable operator T possesses a spectral function E_T, defined on all bounded Borel sets $\Delta \subseteq \mathbb{R}$, which boundary points (in $\mathbb{R}$) are not zeros of every definitizing polynomial, and their complements $\mathbb{R}\backslash\Delta$, see [73, Theorem II.3.1]. Moreover, we have $E_T(\sigma) = E(\sigma, T)$ for a spectral set $\sigma \subseteq \mathbb{R}$ of T. The next proposition shows that the non-real spectrum of T is finite and we find a bound on the length of the Jordan chains of T, see [73, Proposition II.2.1].

Proposition 1.11. *Let T be a definitizable operator in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ with definitizing polynomial p. For $\lambda \in \mathbb{C}$ let $k(\lambda)$ denote its multiplicity as a zero of p ($k(\lambda) := 0$ if $p(\lambda) \neq 0$). Then the non-real spectrum of T consists of a finite number of pairs $\mu, \overline{\mu}$ and each isolated spectral point of T is an eigenvalue of T. Moreover, at isolated spectral points $\mu \in \mathbb{R}$ of T the resolvent $(T-\lambda)^{-1}$ has poles of order at most $k(\mu)+1$. In particular, if $\mu \in \mathbb{R}$ is an eigenvalue of T then each corresponding Jordan chain is of length at most $k(\mu)+1$.*

Note that Proposition 1.11 also shows that the resolvent set of a definitizable operator T is dense in $\mathbb{C}$. Moreover, Weyl's essential spectrum theorem shows, that the essential spectrum

$$\sigma_{\mathrm{ess}}(T) := \{\lambda \in \mathbb{C} \,|\, T - \lambda \text{ is not Fredholm}\}$$

of T is stable under compact additive perturbations in the resolvent sense, see [48, Theorem IX.2.4] (cf. also [48, Proposition IX.2.5]). Note, that for real eigenvalues λ of T with $\dim \mathfrak{L}_\lambda(T) < \infty$ we can decompose the Krein space $\mathcal{K}$ into the root subspace and its orthogonal complement.

Proposition 1.12. *Let T be a definitizable operator in $(\mathcal{K}, [\cdot,\cdot])$ and let $\lambda \in \mathbb{R}$ be an isolated eigenvalue of T with $\dim \mathfrak{L}_\lambda(T) < \infty$. Then $\mathcal{K} = \mathcal{L}_\lambda(T)[\dot{+}]\mathcal{L}_\lambda(T)^{[\perp]}$, where both $(\mathcal{L}_\lambda(T), [\cdot,\cdot])$ and $(\mathcal{L}_\lambda(T)^{[\perp]}, [\cdot,\cdot])$ are Krein spaces.*

Proof. As $\rho(T)$ is dense in $\mathbb{C}$, this follows from [66, §IV.5.6] with [66, Theorem IV.5.28 and §III.6.5]. $\square$

Among the eigenvalues there are two particularly interesting types: A real isolated eigenvalue λ of T is called of *positive* (*negative*) *type* if all its corresponding eigenvectors are positive (negative, respectively), see for example [75]. In this case we write $\lambda \in \sigma_{++}(T)$ ($\lambda \in \sigma_{--}(T)$, respectively). Observe that for an isolated eigenvalue of positive or negative type there is no Jordan chain of length greater than one (see Lemma 1.8), that is, $\mathcal{L}_\lambda(T) = \ker(T-\lambda)$. Furthermore, the resolvent of T has a pole of order one at such a point. This follows from [73, Theorem II.3.1 (4)] and Proposition 1.11.

1.3 Operators with Finitely Many Negative Squares

An important subclass of the class of definitizable operators are the operators with finitely many negative squares, see for example [73, §I.3, Example (c)].

Let T be a selfadjoint operator in $(\mathcal{K}, [\cdot,\cdot])$ with $\rho(T) \neq \emptyset$. We say T has κ *negative squares*, $\kappa \in \mathbb{N}_0$, if the hermitian form $\langle\cdot,\cdot\rangle$ on $\operatorname{dom} T$ defined by

$$\langle f, g\rangle := [Tf, g], \quad f, g \in \operatorname{dom} T,$$

has κ negative squares, that is, there exists a κ-dimensional subspace $\mathcal{M} \subseteq \operatorname{dom} T$ such that $\langle v, v\rangle < 0$ if $v \in \mathcal{M}$, $v \neq 0$, but no $(\kappa+1)$-dimensional subspace with this property. If T has $\kappa = 0$ negative squares, we call T *non-negative.*

In the next theorem we present some spectral properties of operators with finitely many negative squares. The statements are well-known and are consequences of the general results in [72] and [73]. A short proof can be found in [25, Theorem 3.1]. By $\mathbb{C}^+$ ($\mathbb{C}^-$) we denote the open upper (respectively lower) half plane.

Theorem 1.13. *Let T be an operator with κ negative squares in $(\mathcal{K}, [\cdot,\cdot])$. Then the following hold.*

(i) *The non-real spectrum of T consists of at most κ pairs $\{\mu_i, \overline{\mu}_i\}$, $\mu_i \in \mathbb{C}^+$, of eigenvalues with finite-dimensional algebraic eigenspaces. For an eigenvalue λ of T, let $\{\kappa_0(\lambda), \kappa_+(\lambda), \kappa_-(\lambda)\}$ denote the (unique) dimensions of a decomposition of $\mathfrak{L}_\lambda(T)$ as in (1.1). Then*

$$\sum_{\lambda\in\sigma_{\mathrm{p}}(T)\cap(-\infty,0)} \big(\kappa_+(\lambda)+\kappa_0(\lambda)\big) + \sum_{\lambda\in\sigma_{\mathrm{p}}(T)\cap(0,\infty)} \big(\kappa_-(\lambda)+\kappa_0(\lambda)\big) + \sum_i \kappa_0(\mu_i) \le \kappa,$$

and equality holds if $0 \notin \sigma_{\mathrm{p}}(T)$.

(ii) *There are at most κ different real non-zero eigenvalues of T with corresponding Jordan chains of length greater than one. The length of each of these chains is at most $2\kappa+1$.*

(iii) *Let B be a selfadjoint operator in $(\mathcal{K}, [\cdot,\cdot])$ with $\rho(T)\cap\rho(B) \neq \emptyset$ and assume*

$$\dim\big(\operatorname{ran}(T-\lambda)^{-1} - (B-\lambda)^{-1}\big) = n_0 < \infty$$

for some $\lambda \in \rho(T)\cap\rho(B)$. Then B has $\tilde{\kappa} \ge 0$ negative squares, where $|\tilde{\kappa}-\kappa| \le n_0$.

Theorem 1.13(i) states, that there are at most κ isolated eigenvalues which are in a certain sense placed in wrong order, namely there are at most κ eigenvalues λ in $(0,\infty)\cup(-\infty,0)$ which are not of positive type (if $\lambda\in(0,\infty)$) or not of negative type (if $\lambda \in (-\infty,0)$). If $\kappa = 0$ or $\kappa = 1$ this reads as follows, see also [73] and [25, Theorem 3.1].

Corollary 1.14. *Let T be a definitizable operator in $(\mathcal{K}, [\cdot,\cdot])$.*

(i) *If T is non-negative, the isolated positive (negative) eigenvalues of T belong to $\sigma_{++}(T)$ ($\sigma_{--}(T)$, respectively).*

(ii) *If T has one negative square, then there is at most one isolated eigenvalue $\mu \in \mathbb{R}$, $\mu \neq 0$, such that $\mu \notin \sigma_{++}(T)\cap(0,\infty)$ and $\mu \notin \sigma_{--}(T)\cap(-\infty,0)$. Moreover, T has at most one pair of non-real eigenvalues and if such a pair exists, then $\sigma(T)\cap(0,\infty) \subseteq \sigma_{++}(T)$ and $\sigma(T)\cap(-\infty,0)\subseteq\sigma_{--}(T)$.*

Remark 1.15. Let T be a non-negative operator in $(\mathcal{K}, [\cdot,\cdot])$. Then, Theorem 1.13(i) implies $\sigma(T) \subseteq \mathbb{R}$. Moreover, by Proposition 1.11 the Jordan chains of T at 0 are of length at most 2.

Remark 1.16. Let T be an operator with one negative square and assume $\mu \in \mathbb{C}^+$ is an eigenvalue of T. Then $T\restriction(\mathfrak{L}_\mu[\dot{+}]\mathfrak{L}_{\overline{\mu}})^{[\perp]}$ is non-negative.

To show this, note that by [26, Lemma I.10.1 and Theorem VI.6.5] the spaces $\mathfrak{L}_\mu(T)$ and $\mathfrak{L}_{\overline{\mu}}(T)$ are neutral. Moreover, $\mathfrak{L}_\mu(T)[\dot{+}]\mathfrak{L}_{\overline{\mu}}(T)$ is a (2-dimensional) Krein space and there exist $x \in \mathfrak{L}_\mu(T)$ and $y \in \mathfrak{L}_{\overline{\mu}}(T)$ with $[x, y] = 1$. Assume the real part $\mathrm{Re}\mu$ of μ is greater 0. Then

$$[B(\tfrac{1}{\sqrt{2}}(x-y)), \tfrac{1}{\sqrt{2}}(x-y)] = \tfrac{1}{2}[\mu x - \overline{\mu} y, x - y] = -\tfrac{1}{2}\mu[x,y] - \tfrac{1}{2}\overline{\mu}[x,y] = -\mathrm{Re}\mu < 0.$$

Assume there exists $z \in (\mathfrak{L}_\mu [\dot{+}] \mathfrak{L}_{\overline{\mu}})^{[\perp]}$ such that $[Bz, z] < 0$. Then we have $[Bv, v] < 0$, $v \in \mathcal{M}\backslash\{0\}$, for the 2-dimensional space $\mathcal{M} := \mathrm{span}\{y, z\}$, a contradiction. For $\mathrm{Re}\mu < 0$ we choose the vector $\frac{1}{\sqrt{2}}(x+y)$ and the claim follows analogously.

2. Boundary Triplets and Related Classes of Functions

When comparing two selfadjoint operators A and B in a Krein space $(\mathcal{K}, [\cdot,\cdot])$, the symmetric operator $S = A \cap B$ is the part where A and B coincide. Moreover, A and B are two different selfadjoint extensions of S and are therefore connected via a suitable boundary triplet of the adjoint S^+. The corresponding Weyl function relates the spectra of A and B and is one of the main tools used in this thesis. But the operator S is not necessarily densely defined, so that S^+ may only exist as linear relation. Therefore, we will introduce the notions of linear relations, their boundary triplets, Weyl functions, and γ-fields in Section 2.1 and present in Sections 2.2 and 2.3 properties of the Weyl function related to a rank one perturbation of a selfadjoint operator. Finally, we treat the problem of realising Weyl functions in Section 2.4.

2.1 Boundary Triplets, Weyl Functions, and γ-Fields

We recall some basic notions for linear relations. For more detailed information see for example [6, 28, 38, 53]. Let $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space. On $\mathcal{K}^2 := \mathcal{K} \times \mathcal{K}$ we consider the Krein space inner product

$$\big[\{f, f'\}, \{g, g'\}\big] := [f, g] + [f', g'], \qquad \{f, f'\}, \{g, g'\} \in \mathcal{K}^2.$$

A linear subspace T of $\mathcal{K}^2$ (not necessarily closed) will be called *linear relation in* $\mathcal{K}$. The elements $\widehat{f} \in T$ are pairs of the form $\widehat{f} = \{f, f'\}$. For such a linear relation T we use the notation

$$\begin{aligned}
\operatorname{dom}(T) &= \{f \in \mathcal{K} \,|\, \{f, f'\} \in T \text{ for some } f' \in \mathcal{K}\} \text{ is the } \textit{domain} \text{ of } T,\\
\operatorname{ran}(T) &= \{f' \in \mathcal{K} \,|\, \{f, f'\} \in T \text{ for some } f \in \mathcal{K}\} \text{ is the } \textit{range} \text{ of } T,\\
\ker(T) &= \{f \in \mathcal{K} \,|\, \{f, 0\} \in T\} \text{ is the } \textit{kernel} \text{ of } T, \text{ and}\\
\operatorname{mul}(T) &= \{f' \in \mathcal{K} \,|\, \{0, f'\} \in T\} \text{ is the } \textit{multivalued part} \text{ of } T.
\end{aligned}$$

We have the following operations for linear relations T, $S \subseteq \mathcal{K}^2$ and $\alpha \in \mathbb{C}$:

$$\begin{aligned}
\alpha T &= \{\{f, \alpha f'\} \,|\, \{f, f'\} \in T\},\\
T + S &= \{\{f, f' + g'\} \,|\, \{f, f'\} \in T, \{f, g'\} \in S\} \text{ is the } \textit{sum} \text{ of } T \text{ and } S, \text{ and}
\end{aligned}$$

$$TS = \{\{f, f'\} \mid \{f, h\} \in S \text{ and } \{h, f'\} \in T \text{ for some } h \in \mathcal{K}\} \text{ is the } \textit{product} \text{ of } T \text{ and } S.$$

Furthermore, the *inverse* T^{-1} of T always exists and is defined as

$$T^{-1} = \big\{\{f', f\} \mid \{f, f'\} \in T\big\}.$$

In addition to the sum $T + S$ of linear relations, we denote the sum of linear subspaces as

$$T \widehat{+} S = \{\{f + g, f' + g'\} \mid \{f, f'\} \in T, \{g, g'\} \in S\}.$$

If this sum is direct ($T \cap S = \{0\}$), we write $T \widehat{\dot{+}} S$. Moreover, T is *closed* if it is closed as a subspace of $\mathcal{K}^2$. We identify a linear operator T in $\mathcal{K}$ with its *graph* $\operatorname{gr}(T) := \{\{f, Tf\} \mid f \in \operatorname{dom} T\}$. Then the above notions coincide with the corresponding notions for operators. Note, that a linear relation T is an operator if and only if $\operatorname{mul}(T) = \{0\}$. Let $L(\mathcal{K})$ denote the bounded operators $A : \mathcal{K} \to \mathcal{K}$. The *resolvent* set $\rho(T)$ of T consists of the points $\lambda \in \mathbb{C}$ such that $(T - \lambda)^{-1} \in L(\mathcal{K})$. The *spectrum* $\sigma(T)$ of T is its complement: $\sigma(T) = \mathbb{C} \backslash \rho(T)$. The set of *regular type points* of T is given by

$$\widetilde{\rho}(T) := \{\lambda \in \mathbb{C} \mid \ker(T - \lambda) = \{0\}, \operatorname{ran}(T - \lambda) \text{ is closed}\}.$$

Definition 2.1. Let T be a linear relation in $(\mathcal{K}, [\cdot, \cdot])$. The *adjoint relation* (with respect to $[\cdot, \cdot]$) of T is defined by

$$T^+ := \left\{\{g, g'\} \in \mathcal{K}^2 \,\middle|\, [g', f] = [g, f'] \text{ for all } \{f, f'\} \in T\right\}.$$

A linear relation T is said to be *symmetric* (*selfadjoint*) with respect to $[\cdot, \cdot]$, if $T \subseteq T^+$ ($T = T^+$, respectively).

Observe, that this definition extends the usual definition of the adjoint operator (with respect to $[\cdot, \cdot]$) and that $\operatorname{mul}(T^+) = \operatorname{dom}(T)^{[\perp]}$ holds. In particular, T^+ is an operator if and only if T is densely defined. Also, T^+ is closed and $T^{++} = \overline{T}$ holds, see [38, Proposition 3.1], and we easily see $(\operatorname{ran} T)^{[\perp]} = \ker T^+$.

In the following we present the notions of boundary triplets, γ-fields, and Weyl functions for symmetric linear relations, see for example [29, 30, 38, 39] and also [17] in the Krein space situation, and for example [20, 36, 52, 92] in the Hilbert space situation. For the next definition see [30, Definition 2.1]; cf. also [27, Definition 1.1] and [52, Section 3.1.4].

Definition 2.2. Let S be a closed symmetric relation in $(\mathcal{K}, [\cdot, \cdot])$. A triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, consisting of a Hilbert space $(\mathcal{H}, (\cdot, \cdot)_{\mathcal{H}})$ and two linear mappings $\Gamma_0, \Gamma_1 : S^+ \to \mathcal{H}$, is called *boundary triplet* for the relation S^+ if the abstract Green identity

$$[f', g] - [f, g'] = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g})_{\mathcal{H}} - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})_{\mathcal{H}} \tag{2.1}$$

holds for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in S^+$ and the linear mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : S^+ \to \mathcal{H} \times \mathcal{H}$ is surjective.

Let S be a closed symmetric relation in $(\mathcal{K}, [\cdot, \cdot])$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^+. Then it is easy to see that $\Gamma : S^+ \to \mathcal{H} \times \mathcal{H}$ and $\Gamma_0, \Gamma_1 : S^+ \to \mathcal{H}$ are continuous and $\ker\Gamma = S$, see also [37, Remark 1.1]. The set $\widetilde{\rho}(S)$ is symmetric with respect to $\mathbb{R}$, see [30, Section 1.2]. For $\lambda \in \widetilde{\rho}(S)$, the set

$$\mathcal{N}_\lambda(S^+) := \ker(S^+ - \lambda) = \big(\operatorname{ran}(S - \overline{\lambda})\big)^{[\perp]}$$

is the *defect subspace* of S and

$$\widehat{\mathcal{N}}_\lambda(S^+) := \Big\{\{f_\lambda, \lambda f_\lambda\} \,\Big|\, f_\lambda \in \mathcal{N}_\lambda(S^+)\Big\}$$

is the *defect relation*. A linear relation $\widetilde{S} \subseteq \mathcal{K}^2$ with $S \subsetneq \widetilde{S} \subsetneq S^+$ is called *extension* of S. In the next proposition we establish a bijection between the closed extensions of S and the closed linear relations in the boundary space $\mathcal{H}$, see [30, Proposition 2.1].

Proposition 2.3. *Let S be a closed symmetric relation in $(\mathcal{K}, [\cdot,\cdot])$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^+. Then the following statements hold.*

(i) *The mapping $\Gamma = (\Gamma_0, \Gamma_1)^\top$ induces a bijective correspondence between the set of closed extensions A_θ of S and the set of closed linear relations θ in $\mathcal{H}$, via*

$$\theta \mapsto A_\theta := \Big\{\widehat{f} \in S^+ \,\Big|\, \Gamma\widehat{f} \in \theta\Big\}. \tag{2.2}$$

(ii) *The closed extension A_θ of S is symmetric (selfadjoint) in the Krein space $\mathcal{K}$ if and only if the closed relation θ is symmetric (selfadjoint, respectively) in the Hilbert space $\mathcal{H}$.*

There are two special extensions of S,

$$A_0 := \ker\Gamma_0 \quad \text{and} \quad A_1 := \ker\Gamma_1.$$

It is clear that A_0 and A_1 are selfadjoint extensions of S since they correspond to the selfadjoint parameters θ in $\mathcal{H}$ in (2.2) given by

$$\theta_0 = \{0\} \times \mathcal{H} \quad \text{and} \quad \theta_1 = \mathcal{H} \times \{0\}, \quad \text{respectively.} \tag{2.3}$$

If there exists a selfadjoint extension of S with non-empty resolvent set, then a boundary triplet for S^+ exists, see [30, Proposition 2.2].

Proposition 2.4. *Let S be a closed symmetric linear relation in $(\mathcal{K}, [\cdot,\cdot])$ and let $A_0 = A_0^+$ be an extension of S with $\rho(A_0) \neq \emptyset$. Then the following hold.*

(i) *The decomposition*

$$S^+ = A_0 \,\widehat{+}\, \widehat{\mathcal{N}}_\mu(S^+)$$

holds for all $\mu \in \rho(A_0)$.

(ii) *There exists a boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for the relation S^+ such that* $\ker\Gamma_0 = A_0$.

Since $\ker\Gamma_0 = A_0$, we see by Proposition 2.4(i) that $\Gamma_0 \restriction \widehat{\mathcal{N}}_\lambda(S^+) : \widehat{\mathcal{N}}_\lambda(S^+) \to \mathcal{H}$ is injective for $\lambda \in \rho(A_0)$. Furthermore, $\Gamma_0 : S^+ \to \mathcal{H}$ is also surjective and therefore Γ_0 maps $\widehat{\mathcal{N}}_\lambda(S^+)$

bijectively onto $\mathcal{H}$ for $\lambda \in \rho(A_0)$. Consequently, the inverse mapping

$$\lambda \in \rho(A_0), \quad \lambda \mapsto \widehat{\gamma}(\lambda) := (\Gamma_0 \upharpoonright \widehat{\mathcal{N}}_\lambda(S^+))^{-1},$$

maps $\mathcal{H}$ bijectively onto $\widehat{\mathcal{N}}_\lambda(S^+)$ for $\lambda \in \rho(A_0)$. Hence, we see $\dim \mathcal{H} = \dim \widehat{\mathcal{N}}_\lambda(S^+)$ is constant on $\rho(A_0)$. Moreover, S^+/A_0 and $\widehat{\mathcal{N}}_\lambda(S^+)$ are isomorphic, which yields $\dim \mathcal{H} = \dim \widehat{\mathcal{N}}_\lambda(S^+) = \dim(S^+/A_0)$. Additionally, the surjectivity of Γ and $\ker \Gamma = S$ show that S^+/S is isomorphic to $\mathcal{H}^2$ and as $S \subseteq A_0 \subseteq S^+$ we have $\dim(A_0/S) = \dim(S^+/A_0) = \dim \mathcal{H}$.

Let π_i be the projection onto the i-th component of $\mathcal{K}^2$, $i = 1, 2$. Then π_1 maps $\widehat{\mathcal{N}}_\lambda(S^+)$ bijectively onto $\mathcal{N}_\lambda(S^+)$ and the mapping

$$\lambda \mapsto \gamma(\lambda) := \pi_1 \widehat{\gamma}(\lambda), \quad \lambda \in \rho(A_0),$$

is the *γ-field* associated to the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$. We have

$$\begin{aligned} \gamma(\lambda) &= \{\{\Gamma_0 \widehat{f}_\lambda, f_\lambda\} \mid \widehat{f}_\lambda \in \widehat{\mathcal{N}}_\lambda(S^+)\}, && \lambda \in \rho(A_0), \\ \gamma(\lambda) &= \gamma(\omega) + (\lambda - \omega)(A_0 - \lambda)^{-1}\gamma(\omega), && \lambda, \omega \in \rho(A_0), \\ \text{and } \gamma(\overline{\lambda})^+ &= \Gamma_1\{(A_0 - \lambda)^{-1}, I + \lambda(A_0 - \lambda)^{-1}\}, && \lambda \in \rho(A_0), \end{aligned} \tag{2.4}$$

by [30, Proposition 2.2]. Related to a boundary triplet there is also the holomorphic operator-valued *Weyl function*, see [30, Definition 2.2].

Definition 2.5. Let S be a closed symmetric relation in $(\mathcal{K}, [\cdot, \cdot])$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^+. Set $A_0 := \ker \Gamma_0$. The mapping

$$\lambda \mapsto M(\lambda) := \Gamma_1 \big(\Gamma_0 \upharpoonright \widehat{\mathcal{N}}_\lambda(S^+)\big)^{-1} = \big\{\{\Gamma_0 \widehat{f}_\lambda, \Gamma_1 \widehat{f}_\lambda\} \mid \widehat{f}_\lambda \in \widehat{\mathcal{N}}_\lambda(S^+)\big\}, \quad \lambda \in \rho(A_0),$$

is called *Weyl function* associated to the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$.

Some of the basic properties of Weyl functions are collected in the next proposition, cf. [30, Section 2.2 and Proposition 2.3].

Proposition 2.6. *Let S be a closed symmetric relation in $(\mathcal{K}, [\cdot, \cdot])$ and let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^+. Set $A_0 := \ker \Gamma_0$. Then the following statements hold for the corresponding γ-field γ and Weyl function M.*

(i) *$M(\lambda) \in L(\mathcal{H})$ for all $\lambda \in \rho(A_0)$.*

(ii) *For all $\lambda \in \rho(A_0)$ one has $M(\lambda)\Gamma_0 \widehat{f}_\lambda = \Gamma_1 \widehat{f}_\lambda$ for every $\widehat{f}_\lambda \in \widehat{\mathcal{N}}_\lambda(S^+)$.*

(iii) *For all $\lambda, \mu \in \rho(A_0)$ the identity*

$$M(\lambda) - M(\mu)^* = (\lambda - \overline{\mu})\gamma(\mu)^+\gamma(\lambda)$$

holds, and in particular M has the property $M(\lambda)^* = M(\overline{\lambda}), \lambda \in \rho(A_0)$.

The following Krein type resolvent formula shows how the difference of resolvents of two extensions can be expressed in terms of the Weyl function and γ-field, see for example [30, Theorem 2.1].

Theorem 2.7. *Let S be a closed symmetric relation in $(\mathcal{K},[\cdot,\cdot])$, $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ be a boundary triplet for S^+, and $A_0=\ker\Gamma_0$. Let M be the corresponding Weyl function and $\theta\subseteq\mathcal{H}^2$ be a closed linear relation in $\mathcal{H}$. Then $\lambda\in\rho(A_\theta)$ if and only if $0\in\rho(M(\lambda)-\theta)$ and for all $\lambda\in\rho(A_\theta)\cap\rho(A_0)$ the following equality holds*

$$(A_0-\lambda)^{-1}-(A_\theta-\lambda)^{-1}=\gamma(\lambda)(M(\lambda)-\theta)^{-1}\gamma(\overline{\lambda})^+.$$

In general, there are many possible boundary triplets for a symmetric relation (if there exist any). Moreover, any given boundary triplet can be transformed into a new one, cf. for example [36, Proposition 1.7]. One of these transformations is given in the following lemma, which is easily seen.

Lemma 2.8. *Let S be a closed symmetric relation in $(\mathcal{K},[\cdot,\cdot])$ and assume that $\{\mathcal{H},\Gamma_0,\Gamma_1\}$ is a boundary triplet for S^+ with γ-field γ and Weyl function M. Define*

$$\underline{\Gamma}_0:=\Gamma_1\quad\textit{and}\quad\underline{\Gamma}_1:=-\Gamma_0.$$

Then $\{\mathcal{H},\underline{\Gamma}_0,\underline{\Gamma}_1\}$ is a boundary triplet for S^+ and $\ker\underline{\Gamma}_0=\ker\Gamma_1$. For $\lambda\in\rho(\ker\Gamma_0)\cap\rho(\ker\Gamma_1)$ the corresponding γ-field $\underline{\gamma}$ and Weyl function $\underline{M}$ are given by

$$\underline{\gamma}(\lambda)=\gamma(\lambda)M(\lambda)^{-1},\quad \underline{M}(\lambda)=-M(\lambda)^{-1},$$

and $M(\lambda)$ is an isomorphism on $\mathcal{H}$.

2.2 Boundary Triplets, Weyl Functions, and Rank One Perturbations

In this section we explore how the spectral properties of two one-dimensional extensions of a symmetric operator are encoded in the corresponding Weyl function. In the case of operators with finitely many negative squares, this topic was studied by R. Möws in [86], cf. also [21, 22]. In the case of rank one perturbations these results were extended in [17], where the results of this section can be found.

In the following let A and B be selfadjoint operators in the Krein space $(\mathcal{K},[\cdot,\cdot])$ such that $\rho(A)\cap\rho(B)\neq\emptyset$ and

$$\dim\operatorname{ran}\left((A-\lambda_0)^{-1}-(B-\lambda_0)^{-1}\right)=1 \tag{2.5}$$

holds for some (and hence for all) $\lambda_0\in\rho(A)\cap\rho(B)$. Then we find non-zero vectors $e,f\in\mathcal{K}$ such that

$$(A-\lambda_0)^{-1}-(B-\lambda_0)^{-1}=[\cdot,e]f. \tag{2.6}$$

The operator $S:=A\cap B$, that is

$$\operatorname{dom}S=\{f\in\operatorname{dom}A\cap\operatorname{dom}B\mid Af=Bf\},\quad Sf=Af=Bf,$$

is a (in general non-densely defined) closed symmetric operator in $\mathcal{K}$. The adjoint S^+ is an operator if and only if S is densely defined and a linear relation otherwise. For simplicity, we understand S^+ as linear relation in either case. It is easy to check that

$$(S-\lambda_0)^{-1} = (A-\lambda_0)^{-1} \cap (B-\lambda_0)^{-1} \tag{2.7}$$

holds, where the intersection (2.7) is understood in the sense of linear relations. Hence (2.6) and (2.7) imply $\dim\big(\operatorname{ran}(A-\lambda_0)^{-1}/\operatorname{ran}(S-\lambda_0)^{-1}\big) = \dim(\operatorname{dom} A/\operatorname{dom} S) = 1$. Therefore, $\dim(A/S) = 1$.

The following proposition provides again a Krein type formula. But in contrast to Theorem 2.7 we start with the two selfadjoint extensions A and B of S and construct a special boundary triplet for S^+. This result is an easy modification of [20, Corollary 2.5], which holds in the Hilbert space case, cf. also [35] and [76].

Proposition 2.9. *Let A and B be selfadjoint operators in $(\mathcal{K},[\cdot,\cdot])$ such that $\rho(A)\cap\rho(B) \neq \emptyset$ and* (2.5) *holds for some* (*and hence for all*) $\lambda_0 \in \rho(A)\cap\rho(B)$. *Then $S = A\cap B$ is a closed symmetric operator in $\mathcal{K}$ and there exists a boundary triplet $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ for S^+ such that $A = \ker\Gamma_0$ and $B = \ker\Gamma_1$ hold. Moreover, if γ and M denote the γ-field and Weyl function corresponding to $\{\mathbb{C},\Gamma_0,\Gamma_1\}$, then*

$$(A-\lambda)^{-1} - (B-\lambda)^{-1} = \gamma(\lambda)M(\lambda)^{-1}\gamma(\overline{\lambda})^+ = \frac{1}{M(\lambda)}[\cdot,\gamma(\overline{\lambda})]\gamma(\lambda)$$

holds for all $\lambda \in \rho(A)\cap\rho(B)$.

Proof. We fix a fundamental symmetry J of $\mathcal{K}$ and denote the corresponding Hilbert space product on $\mathcal{K}^2$ by $(\cdot,\cdot)$. Let $\lambda \in \rho(A)\cap\rho(B)$. Since $\dim(A/S) = 1$, we can represent elements $\widehat{f} \in S^+$ by Proposition 2.4 as

$$\widehat{f} = \{f,f'\} = \{f_A + f_\lambda, Af_A + \lambda f_\lambda\}, \quad f_A \in \operatorname{dom} A,\ f_\lambda \in \mathcal{N}_\lambda(S^+). \tag{2.8}$$

Let P_0 denote the $(\cdot,\cdot)$-orthogonal projection onto the closed subspace $\mathcal{N}_\lambda(S^+)$. We define a triplet $\{\mathcal{N}_\lambda(S^+),\underline{\Gamma}_0,\underline{\Gamma}_1\}$ via

$$\underline{\Gamma}_0,\underline{\Gamma}_1 : S^+ \to \mathcal{N}_\lambda(S^+), \quad \underline{\Gamma}_0\widehat{f} := f_\lambda \quad \text{and} \quad \underline{\Gamma}_1\widehat{f} := P_0J\left((A-\overline{\lambda})f_A + \lambda f_\lambda\right).$$

Then, $\{\mathbb{C},\underline{\Gamma}_0,\underline{\Gamma}_1\}$ is a boundary triplet for S^+. To see this, let $\widehat{f} \in S^+$ as in (2.8) and $\widehat{g} = \{g,g'\} = \{g_A + g_\lambda, Ag_A + \lambda g_\lambda\}$. We obtain from $[Af_A,g_A] = [f_A,Ag_A]$ that

$$\begin{aligned}
[f',g] - [f,g'] &= [Af_A + \lambda f_\lambda, g_A + g_\lambda] - [f_A + f_\lambda, Ag_A + \lambda g_\lambda] \\
&= [(A-\overline{\lambda})f_A + \lambda f_\lambda, g_\lambda] - [f_\lambda, (A-\overline{\lambda})g_A + \lambda g_\lambda] \\
&= \big(J\big((A-\overline{\lambda})f_A + \lambda f_\lambda\big), g_\lambda\big) - \big(f_\lambda, J\big((A-\overline{\lambda})g_A + \lambda g_\lambda\big)\big) \\
&= (\underline{\Gamma}_1\widehat{f},\underline{\Gamma}_0\widehat{g}) - (\underline{\Gamma}_0\widehat{f},\underline{\Gamma}_1\widehat{g})
\end{aligned}$$

holds. Since $\overline{\lambda} \in \rho(A)$, for $x,y \in \mathcal{N}_\lambda(S^+)$ there exists $f_A \in \operatorname{dom} A$ such that $(A-\overline{\lambda})f_A =$

$J(y-\lambda x)$. If we set $\widehat{f} := \{f_A + x, Af_A + \lambda x\}$, we obtain

$$\underline{\Gamma}\widehat{f} = \begin{pmatrix} \underline{\Gamma}_0\widehat{f} \\ \underline{\Gamma}_1\widehat{f} \end{pmatrix} = \begin{pmatrix} x \\ P_0 J\big((A-\overline{\lambda})f_A + \lambda x\big) \end{pmatrix} = \begin{pmatrix} x \\ P_0(y - \lambda x + \lambda x) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, $\underline{\Gamma} : S^+ \to \mathcal{N}_\lambda(S^+) \times \mathcal{N}_\lambda(S^+)$ is surjective. Moreover, $\underline{\Gamma}_0\widehat{f} = f_\lambda = 0$ if and only if $\{f, f'\} = \{f_A, Af_A\} \in A$. Thus, $\ker \underline{\Gamma}_0 = A$ and as $\dim \mathcal{N}_\lambda(S^+) = 1$, we can identify $\mathcal{N}_\lambda(S^+)$ with $\mathbb{C}$. Since B is a selfadjoint extension of the symmetric operator S, $B \subseteq S^+ = \operatorname{dom} \underline{\Gamma}$ and

$$D := \underline{\Gamma}B = \big\{\{\underline{\Gamma}_0\widehat{f}, \underline{\Gamma}_1\widehat{f}\} \,\big|\, f \in \operatorname{dom} B, \widehat{f} = \{f, Bf\}\big\}$$

is a linear relation in $\mathbb{C}$. Then $B = A_D = \{\widehat{f} \in S^+ \,|\, \underline{\Gamma}\widehat{f} \in D\}$ and therefore D is selfadjoint by Proposition 2.3(ii). We claim that D is a (real) constant, i.e. $\operatorname{mul}(D) = \{0\}$. Indeed, if D is multivalued, then $D = \{0\} \times \mathbb{C}$. By (2.3) we then have $B = \ker \underline{\Gamma}_0 = A$, which is a contradiction to (2.5). As D is a constant, $\underline{\Gamma}\widehat{f} \in D$ is equivalent to $\underline{\Gamma}_1\widehat{f} = D\underline{\Gamma}_0\widehat{f}$, which yields $B = \ker(\underline{\Gamma}_1 - D\underline{\Gamma}_0)$.

We set $\Gamma_0 := \underline{\Gamma}_0$ und $\Gamma_1 := \underline{\Gamma}_1 - D\underline{\Gamma}_0$. Then, we have for $\widehat{f}, \widehat{g} \in S^+$

$$\begin{aligned}(\Gamma_1\widehat{f}, \Gamma_0\widehat{g}) - (\Gamma_0\widehat{f}, \Gamma_1\widehat{g}) &= (\underline{\Gamma}_1\widehat{f} - D\underline{\Gamma}_0\widehat{f}, \underline{\Gamma}_0\widehat{g}) - (\underline{\Gamma}_0\widehat{f}, \underline{\Gamma}_1\widehat{g} - D\underline{\Gamma}_0\widehat{g}) \\ &= (\underline{\Gamma}_1\widehat{f}, \underline{\Gamma}_0\widehat{g}) - (\underline{\Gamma}_0\widehat{f}, \underline{\Gamma}_1\widehat{g}) \\ &= [f', g] - [f, g'],\end{aligned}$$

which shows the abstract Green identity. Moreover, the surjectivity of $\underline{\Gamma}$ yields $\underline{\Gamma}_1 \neq D\underline{\Gamma}_0$. Hence, $\Gamma_1 \neq 0$ and $(\Gamma_0, \Gamma_1)^\top : S^+ \to \mathbb{C}^2$ is also surjective. By construction, we have $\ker \Gamma_0 = \ker \underline{\Gamma}_0 = A$ and $\ker \Gamma_1 = \ker(\underline{\Gamma}_1 - D\underline{\Gamma}_0) = B$.

Denote by γ and M ($\underline{\gamma}$ and $\underline{M}$) the γ-field and Weyl function corresponding to the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ ($\{\mathbb{C}, \underline{\Gamma}_0, \underline{\Gamma}_1\}$, respectively). For $\lambda \in \rho(A) \cap \rho(B)$ it follows

$$\begin{aligned}\gamma(\lambda) &= \pi_1\big(\Gamma_0 \restriction \widehat{\mathcal{N}}_\lambda(S^+)\big)^{-1} = \pi_1\big(\underline{\Gamma}_0 \restriction \widehat{\mathcal{N}}_\lambda(S^+)\big)^{-1} = \underline{\gamma}(\lambda) \quad \text{and} \\ M(\lambda) &= \Gamma_1\widehat{\gamma}(\lambda) = \underline{\Gamma}_1\widehat{\gamma}(\lambda) - D\underline{\Gamma}_0\widehat{\gamma}(\lambda) = \underline{M}(\lambda) - D\Gamma_0\big(\Gamma_0 \restriction \widehat{\mathcal{N}}_\lambda(S^+)\big)^{-1} = \underline{M}(\lambda) - D.\end{aligned}$$

Applying Theorem 2.7 to $\{\mathbb{C}, \underline{\Gamma}_0, \underline{\Gamma}_1\}$ we finally see

$$\begin{aligned}(A-\lambda)^{-1} - (B-\lambda)^{-1} &= \underline{\gamma}(\lambda)\,(\underline{M}(\lambda) - D)^{-1}\,\underline{\gamma}(\overline{\lambda})^+ = \gamma(\lambda)M(\lambda)^{-1}\gamma(\overline{\lambda})^+ \\ &= \frac{1}{M(\lambda)}[\cdot, \gamma(\overline{\lambda})]\gamma(\lambda)\end{aligned}$$

for all $\lambda \in \rho(A) \cap \rho(B)$. The last assertion holds since the boundary value space $\mathcal{H} = \mathbb{C}$ is one dimensional. □

Note that in the situation of Proposition 2.9 $M : \mathbb{C} \to \mathbb{C}$ is a scalar function. The following proposition can be deduced from similar considerations as in [11, 77], see also [17]. Here we give a direct proof based on the previous statements.

Proposition 2.10. *Let A and B be selfadjoint operators in $(\mathcal{K}, [\cdot,\cdot])$ which satisfy* (2.5) *for some $\lambda_0 \in \rho(A) \cap \rho(B)$. Then there exist holomorphic functions $M_A : \rho(A) \to \mathbb{C}$, $M_B : \rho(B) \to \mathbb{C}$ symmetric with respect to $\mathbb{R}$ and vectors φ_A, φ_B in $\mathcal{K}$ such that the following holds.*

(i) *For $\gamma_A(\lambda) := (1 + (\lambda - \lambda_0)(A - \lambda)^{-1})\varphi_A$, $\lambda \in \rho(A)$, we have*

$$M_A(\lambda) - M_A(\overline{\omega}) = (\lambda - \overline{\omega})[\gamma_A(\lambda), \gamma_A(\omega)], \quad \lambda, \omega \in \rho(A).$$

(ii) *For $\gamma_B(\lambda) := (1 + (\lambda - \lambda_0)(B - \lambda)^{-1})\varphi_B$, $\lambda \in \rho(B)$, we have*

$$M_B(\lambda) - M_B(\overline{\omega}) = (\lambda - \overline{\omega})[\gamma_B(\lambda), \gamma_B(\omega)], \quad \lambda, \omega \in \rho(B).$$

(iii) *For $\lambda \in \rho(A) \cap \rho(B)$ we have $M_B(\lambda) = -\frac{1}{M_A(\lambda)}$ and*

$$(A - \lambda)^{-1} - (B - \lambda)^{-1} = \frac{1}{M_A(\lambda)}[\cdot, \gamma_A(\overline{\lambda})]\gamma_A(\lambda) = -\frac{1}{M_B(\lambda)}[\cdot, \gamma_B(\overline{\lambda})]\gamma_B(\lambda).$$

Proof. Consider $S = A \cap B$ and let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for the adjoint S^+ such that $A = \ker \Gamma_0$ and $B = \ker \Gamma_1$ as in Proposition 2.9. Let γ and M be the corresponding γ-field and Weyl function, and define $\varphi_A := \gamma(\lambda_0)$. From the property (2.4) we see that $\gamma_A = \gamma$ holds. Moreover, $M_A := M$ satisfies the formula in (i), see again (2.4). Observe that by Lemma 2.8 $\{\mathbb{C}, \Gamma_1, -\Gamma_0\}$ is a boundary triplet for S^+. Let $\underline{\gamma}$ and $\underline{M}$ be the corresponding γ-field and Weyl function and define $\varphi_B := \underline{\gamma}(\lambda_0)$. As above it follows that $\gamma_B = \underline{\gamma}$. The function $M_B := \underline{M}$ satisfies the assertion in (ii). By Lemma 2.8 we also have $\underline{M}(\lambda) = -M(\lambda)^{-1}$ and hence $M_B(\lambda) = -M_A(\lambda)^{-1}$, $\lambda \in \rho(A) \cap \rho(B)$, as stated in (iii). Finally, the remaining resolvent formula in (iii) follows from Proposition 2.9 applied to $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for the first equality and to $\{\mathbb{C}, \Gamma_1, -\Gamma_0\}$ for the second equality. □

Corollary 2.11. *Let A, B and M_A, M_B be as in Proposition 2.10. Then the following holds.*

(i) *For $\lambda \in \rho(A)$ we have $\lambda \in \sigma_{\mathrm{p}}(B)$ if and only if $M_A(\lambda) = 0$.*

(ii) *For $\lambda \in \rho(B)$ we have $\lambda \in \sigma_{\mathrm{p}}(A)$ if and only if $M_B(\lambda) = 0$.*

Proof. (i) Since the functions γ_A and M_A are holomorphic in a neighbourhood of $\lambda \in \rho(A)$, this follows from the resolvent formula in Proposition 2.10(iii). Assertion (ii) follows in a similar way. □

2.3 Rank One Perturbations and Interlacing of Eigenvalues

In this section we analyse the properties of the complex-valued functions M_A and M_B of Proposition 2.10 under the additional Assumption I below. We explore how the spectral properties of A and B are encoded in these functions, which among other things leads to conclusions about the interlacing of eigenvalues, cf. Proposition 2.15. The results of this section can be found in [17], see also [86].

Assumption I. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that (2.5) holds for some (and hence for all) $\lambda_0 \in \rho(A) \cap \rho(B)$ with functions M_A and M_B as in Proposition 2.10. Let $I \subseteq \mathbb{R}$ be an open interval and assume that $\rho(B) \cap I \neq \emptyset$ and $\sigma(A) \cap I$ consists only of isolated eigenvalues which are poles of the resolvent of A.*

Assumption I yields the following statements.

Theorem 2.12. *Let A, B, and I be as in Assumption* I.

(i) *If $\mu \in \rho(A) \cap I$ then either $\mu \in \rho(B)$ or $\mu \in \sigma_{\mathrm{p}}(B)$ with* $\dim\ker(B-\mu) = 1$. *If, in addition, $\mu \in \sigma_{\pm\pm}(B)$ then $\mathcal{L}_\mu(B) = \ker(B-\mu)$.*

(ii) *If $\mu \in \rho(B) \cap I$ then either $\mu \in \rho(A)$ or $\mu \in \sigma_{\mathrm{p}}(A)$ with* $\dim\ker(A-\mu) = 1$. *If, in addition, $\mu \in \sigma_{\pm\pm}(A)$ then $\mathcal{L}_\mu(A) = \ker(A-\mu)$.*

Proof. Let $\mu \in \rho(A)$ and let $S = A \cap B$ as in Section 2.2. Assume $\dim\ker(B-\mu) > 1$. Then there exist 2 linearly independent vectors $\{x_1, x_2\}$ in $\ker(B-\mu)$. In view of (2.6) there also exist $\alpha_1, \alpha_2 \in \mathbb{C}$, such that $z := \alpha_1(B-\lambda_0)x_1 + \alpha_2(B-\lambda_0)x_2$ fulfils

$$(A-\lambda_0)^{-1}z - (B-\lambda_0)^{-1}z = [z,e]f = 0.$$

Consequently, $y := (A-\lambda_0)^{-1}z = (B-\lambda_0)^{-1}z$ and $(B-\mu)y = (B-\mu)(B-\lambda_0)^{-1}(B-\lambda_0)(\alpha_1 x_1 + \alpha_2 x_2) = 0$. It follows

$$(A-\mu)y = (A-\mu)(A-\lambda_0)^{-1}z = z + (\lambda_0-\mu)y = (B-\mu)(B-\lambda_0)^{-1}z = (B-\mu)y = 0,$$

but $y \neq 0$, a contradiction. Hence, $\dim\ker(B-\mu) \leq 1$. Eigenvectors with a Jordan chain of length greater than one are neutral (cf. Lemma 1.8) and hence (i) is shown. Statement (ii) is proved analogously. □

In the next lemma we relate sign type properties of eigenvalues of B in $\rho(A)$ with the local behaviour of the function M_A from Proposition 2.10, see also [80, Theorem 3.3].

Lemma 2.13. *Let A, B, and I be as in Assumption* I. *Assume $M_A(\mu) = 0$ for some $\mu \in \rho(A) \cap I$. Then $\mu \in \sigma_{\mathrm{p}}(B)$ and* $\dim\ker(B-\mu) = 1$. *Moreover, the following assertions hold.*

(i) *$\mu \in \sigma_{++}(B)$ if and only if $M_A'(\mu) > 0$. In this case $\mathcal{L}_\mu(B) = \ker(B-\mu)$.*

(ii) *$\mu \in \sigma_{--}(B)$ if and only if $M_A'(\mu) < 0$. In this case $\mathcal{L}_\mu(B) = \ker(B-\mu)$.*

(iii) *$\mu \in \sigma_{\mathrm{p}}(B)$ has a neutral eigenvector if and only if $M_A'(\mu) = 0$. In this case $\mathcal{L}_\mu(B) \neq \ker(B-\mu)$ and there exist non-zero elements $x_0 \in \ker(B-\mu)$, $x_1 \in \mathcal{L}_\mu(B)$ with $(B-\mu)x_1 = x_0$ and $(B-\mu)x_0 = 0$ such that*

$$[x_0, x_0] = M_A'(\mu) = 0 \quad \text{and} \quad [x_1, x_0] = \frac{1}{2}M_A''(\mu). \tag{2.9}$$

Moreover, in this case, $(\mathcal{L}_\mu(B), [\cdot,\cdot])$ is a Krein space with at least one positive and one negative element.

Proof. By Corollary 2.11 $M_A(\mu)=0$ implies $\mu\in\sigma_{\mathrm{p}}(B)$ and $\dim\ker(B-\mu)=1$ follows from Theorem 2.12(i). In order to show (i)–(iii) we start with the following observation. For M_A, φ_B, γ_B as in Proposition 2.10 and $\lambda\in\rho(A)\cap\rho(B)$ we conclude from Proposition 2.10(iii)

$$\begin{aligned}M_A(\lambda)\gamma_B(\lambda)&=M_A(\lambda)\big(1+(\lambda-\lambda_0)(B-\lambda)^{-1}\big)\varphi_B\\&=M_A(\lambda)\Big(\varphi_B+(\lambda-\lambda_0)\Big((A-\lambda)^{-1}\varphi_B-\frac{1}{M_A(\lambda)}[\varphi_B,\gamma_A(\overline{\lambda})]\gamma_A(\lambda)\Big)\Big)\\&=(\lambda_0-\lambda)[\varphi_B,\gamma_A(\overline{\lambda})]\gamma_A(\lambda)+M_A(\lambda)\left(1+(\lambda-\lambda_0)(A-\lambda)^{-1}\right)\varphi_B.\end{aligned}\tag{2.10}$$

Then $M_A(\mu)=0$ and $\mu\in\rho(A)\cap\mathbb{R}$ imply the existence of

$$x_0:=\lim_{\lambda\to\mu}M_A(\lambda)\gamma_B(\lambda)=(\lambda_0-\mu)[\varphi_B,\gamma_A(\mu)]\gamma_A(\mu).$$

The vector x_0 is non-zero. Indeed, for $\omega\in\rho(A)\cap\rho(B)$, $\overline{\omega}\neq\mu$, it follows from Proposition 2.10 that

$$\begin{aligned}[x_0,\gamma_B(\omega)]&=\lim_{\lambda\to\mu}[M_A(\lambda)\gamma_B(\lambda),\gamma_B(\omega)]=\lim_{\lambda\to\mu}M_A(\lambda)\frac{M_B(\lambda)-M_B(\overline{\omega})}{\lambda-\overline{\omega}}\\&=\lim_{\lambda\to\mu}M_A(\lambda)\frac{-\frac{1}{M_A(\lambda)}+\frac{1}{M_A(\overline{\omega})}}{\lambda-\overline{\omega}}=\lim_{\lambda\to\mu}\frac{-1+\frac{M_A(\lambda)}{M_A(\overline{\omega})}}{\lambda-\overline{\omega}}=\frac{-1}{\mu-\overline{\omega}}\neq 0.\end{aligned}$$

Furthermore $x_0\in\ker(B-\mu)$, since for $\omega\in\rho(B)$ we have

$$\begin{aligned}(B-\omega)^{-1}x_0&=\lim_{\lambda\to\mu}(B-\omega)^{-1}M_A(\lambda)\gamma_B(\lambda)\\&=\lim_{\lambda\to\mu}(B-\omega)^{-1}M_A(\lambda)\big(1+(\lambda-\lambda_0)(B-\lambda)^{-1}\big)\varphi_B\\&=\lim_{\lambda\to\mu}\frac{M_A(\lambda)}{\lambda-\omega}\Big((\lambda-\omega)(B-\omega)^{-1}+(\lambda-\lambda_0)(B-\lambda)^{-1}-(\lambda-\lambda_0)(B-\omega)^{-1}\Big)\varphi_B\\&=\lim_{\lambda\to\mu}\frac{M_A(\lambda)}{\lambda-\omega}\left((\lambda-\lambda_0)(B-\lambda)^{-1}-(\omega-\lambda_0)(B-\omega)^{-1}\right)\varphi_B\\&=\lim_{\lambda\to\mu}\frac{M_A(\lambda)}{\lambda-\omega}\left(\gamma_B(\lambda)-\gamma_B(\omega)\right)=\frac{1}{\mu-\omega}x_0.\end{aligned}\tag{2.11}$$

Proposition 2.10(ii) and (iii) imply

$$\begin{aligned}[x_0,x_0]&=\lim_{\lambda,\omega\to\mu}M_A(\lambda)\overline{M_A(\omega)}[\gamma_B(\lambda),\gamma_B(\omega)]=\lim_{\lambda,\omega\to\mu}M_A(\lambda)M_A(\overline{\omega})\frac{-\frac{1}{M_A(\lambda)}+\frac{1}{M_A(\overline{\omega})}}{\lambda-\overline{\omega}}\\&=\lim_{\lambda,\omega\to\mu}\frac{M_A(\lambda)-M_A(\overline{\omega})}{\lambda-\overline{\omega}}=\lim_{\lambda\to\mu}\frac{M_A(\lambda)-M_A(\mu)}{\lambda-\mu}=M_A'(\mu).\end{aligned}$$

This yields (i), (ii), and the first statement in (iii). In order to show the remaining statements of (iii) assume $M_A(\mu)=M_A'(\mu)=0$. Relation (2.10) implies the existence of

$$\begin{aligned}x_1&:=\lim_{\lambda\to\mu}\left(M_A(\lambda)\gamma_B(\lambda)\right)'\\&=-[\varphi_B,\gamma_A(\overline{\mu})]\gamma_A(\mu)+(\lambda_0-\mu)[\varphi_B,\gamma_A'(\overline{\mu})]\gamma_A(\mu)+(\lambda_0-\mu)[\varphi_B,\gamma_A(\overline{\mu})]\gamma_A'(\mu).\end{aligned}$$

We obtain

$$
\begin{aligned}
(B-\omega)^{-1}x_1 &= \lim_{\lambda\to\mu}(B-\omega)^{-1}\left(M_A(\lambda)\gamma_B(\lambda)\right)' \\
&= \lim_{\lambda\to\mu}\left((B-\omega)^{-1}M_A'(\lambda)\gamma_B(\lambda)+(B-\omega)^{-1}M_A(\lambda)\gamma_B'(\lambda)\right).
\end{aligned}
\tag{2.12}
$$

As in (2.11) one verifies

$$
(B-\omega)^{-1}M_A'(\lambda)\gamma_B(\lambda)=\frac{M_A'(\lambda)}{\lambda-\omega}\left(\gamma_B(\lambda)-\gamma_B(\omega)\right)
$$

and we have from Proposition 2.10(ii) $\gamma_B'(\lambda)=(B-\lambda)^{-1}\gamma_B(\lambda)$. Hence (2.12) takes the form

$$
(B-\omega)^{-1}x_1=\lim_{\lambda\to\mu}\left(\frac{M_A'(\lambda)}{\lambda-\omega}\left(\gamma_B(\lambda)-\gamma_B(\omega)\right)+(B-\omega)^{-1}M_A(\lambda)(B-\lambda)^{-1}\gamma_B(\lambda)\right)
$$

and with $M_A'(\mu)=0$ we conclude

$$
\begin{aligned}
(B-\omega)^{-1}x_1 &= \lim_{\lambda\to\mu}\left(\frac{M_A'(\lambda)\gamma_B(\lambda)}{\lambda-\omega}+\frac{M_A(\lambda)\gamma_B'(\lambda)}{\lambda-\omega}-(B-\omega)^{-1}\frac{M_A(\lambda)\gamma_B(\lambda)}{\lambda-\omega}\right) \\
&= \lim_{\lambda\to\mu}\left(\frac{\left(M_A(\lambda)\gamma_B(\lambda)\right)'}{\lambda-\omega}-(B-\omega)^{-1}\frac{M_A(\lambda)\gamma_B(\lambda)}{\lambda-\omega}\right) \\
&= \frac{x_1}{\mu-\omega}-(B-\omega)^{-1}\frac{x_0}{\mu-\omega}=\frac{x_1}{\mu-\omega}-\frac{x_0}{(\mu-\omega)^2}.
\end{aligned}
$$

This yields $(B-\mu)x_1=x_0$. Moreover, Proposition 2.10(ii) and (iii) imply

$$
\begin{aligned}
[x_1,x_0] &= \lim_{\lambda,\omega\to\mu}\left[\left(M_A(\lambda)\gamma_B(\lambda)\right)',M_A(\omega)\gamma_B(\omega)\right]=\lim_{\lambda,\omega\to\mu}\frac{d}{d\lambda}\left[M_A(\lambda)\gamma_B(\lambda),M_A(\omega)\gamma_B(\omega)\right] \\
&= \lim_{\lambda,\omega\to\mu}\frac{d}{d\lambda}\left(M_A(\lambda)M_A(\overline{\omega})\frac{-\frac{1}{M_A(\lambda)}+\frac{1}{M_A(\overline{\omega})}}{\lambda-\overline{\omega}}\right)=\lim_{\lambda,\omega\to\mu}\frac{d}{d\lambda}\left(\frac{M_A(\lambda)-M_A(\overline{\omega})}{\lambda-\overline{\omega}}\right) \\
&= \lim_{\lambda\to\mu}\frac{d}{d\lambda}\left(\frac{M_A(\lambda)}{\lambda-\mu}\right)=\lim_{\lambda\to\mu}\left(\frac{M_A'(\lambda)(\lambda-\mu)-M_A(\lambda)}{(\lambda-\mu)^2}\right)=\frac{1}{2}M_A''(\mu),
\end{aligned}
$$

where the last equality follows from the power series expansion of M_A in μ and $M_A(\mu)=M_A'(\mu)=0$. By Propositions 1.9 and 1.7, the space $(\mathcal{L}_\mu(B),[\cdot,\cdot])$ is a Krein space and (iii) is shown. □

Lemma 2.14. *Let A, B, and I be as in Assumption* I *and let $\mu\in I\cap\sigma_{++}(A)$ ($\mu\in I\cap\sigma_{--}(A)$) with $\mu\in\rho(B)$. Then the function M_A has a pole at μ of order one with*

$$
\begin{gathered}
\lim_{\lambda\nearrow\mu}M_A(\lambda)=+\infty,\quad \lim_{\lambda\searrow\mu}M_A(\lambda)=-\infty \\
\left(\lim_{\lambda\nearrow\mu}M_A(\lambda)=-\infty,\quad \lim_{\lambda\searrow\mu}M_A(\lambda)=+\infty,\ \textit{respectively}\right).
\end{gathered}
$$

Proof. According to Theorem 2.12 $\mathcal{L}_\mu(A)=\ker(A-\mu)$ is a one-dimensional subspace.

The corresponding Riesz-Dunford projection onto $\ker(A-\mu)$ will be denoted by E. By Proposition 2.10(i) we have $\gamma_A(\lambda_0)=\varphi_A$ and

$$\begin{aligned} M_A(\lambda) &= M_A(\bar\lambda_0) + (\lambda-\bar\lambda_0)\big[(1+(\lambda-\lambda_0)(A-\lambda)^{-1})\varphi_A,\varphi_A\big] \\ &= M_A(\bar\lambda_0) + (\lambda-\bar\lambda_0)[\varphi_A,\varphi_A] + (\lambda-\bar\lambda_0)(\lambda-\lambda_0)\big[(A-\lambda)^{-1}\varphi_A,\varphi_A\big] \end{aligned}$$

holds for all $\lambda\in\rho(A)$. Since $[E\varphi_A,(I-E)\varphi_A]=0$ and the function

$$\lambda\mapsto\big[(A-\lambda)^{-1}(I-E)\varphi_A,(I-E)\varphi_A\big]$$

is holomorphic in a neighbourhood of the isolated eigenvalue μ we conclude that M_A can be written in the form

$$\begin{aligned} M_A(\lambda) &= h(\lambda) + (\lambda-\bar\lambda_0)(\lambda-\lambda_0)\big[(A-\lambda)^{-1}E\varphi_A,E\varphi_A\big] \\ &= h(\lambda) + \frac{(\lambda-\bar\lambda_0)(\lambda-\lambda_0)}{\mu-\lambda}\big[E\varphi_A,E\varphi_A\big], \end{aligned} \tag{2.13}$$

where h is holomorphic in a neighbourhood of the point μ. In the last equality, we also used

$$(A-\lambda)E\varphi_A = (A-\mu)E\varphi_A + (\mu-\lambda)E\varphi_A = (\mu-\lambda)E\varphi_A,$$

i.e. $(A-\lambda)^{-1}E\varphi_A = (\mu-\lambda)^{-1}E\varphi_A$.

Since by assumption $\mu\in\rho(B)$ we conclude from Corollary 2.11(ii) that the function $M_B = -M_A^{-1}$ has a zero at the point μ, that is, M_A has a pole at μ. As h is holomorphic we obtain $[E\varphi_A,E\varphi_A]\neq 0$ from (2.13). Assume now that $\mu\in\sigma_{++}(A)$ ($\mu\in\sigma_{--}(A)$). Then $[E\varphi_A,E\varphi_A]>0$ ($[E\varphi_A,E\varphi_A]<0$, respectively) and the statements in Lemma 2.14 follow from the representation (2.13). □

The Lemmas 2.13 and 2.14 lead to the following interlacing of eigenvalues of A and B.

Proposition 2.15. *Let A, B, and I be as in Assumption* I. *Let $\mu_1,\mu_2\in\rho(B)\cap I$ such that $(\mu_1,\mu_2)\subseteq\rho(A)$.*

(i) *If $\mu_1,\mu_2\in\sigma_{++}(A)$, then there exists $\mu\in(\mu_1,\mu_2)$ with $\mu\in\sigma_{\mathrm{p}}(B)\backslash\sigma_{--}(B)$.*

(ii) *If $\mu_1,\mu_2\in\sigma_{--}(A)$, then there exists $\mu\in(\mu_1,\mu_2)$ with $\mu\in\sigma_{\mathrm{p}}(B)\backslash\sigma_{++}(B)$.*

Proof. (i) The function M_A has poles of order one at μ_1, μ_2 and its behaviour near these poles is given by Lemma 2.14. Therefore, as M_A is a holomorphic function on $\rho(A)$, it is continuous on $(\mu_1,\mu_2)\subseteq\rho(A)$ and by the intermediate value theorem there exists $\mu\in(\mu_1,\mu_2)$ with $M_A(\mu)=0$ and $M_A'(\mu)\geq 0$. Hence (i) follows from Lemma 2.13. Statement (ii) is shown analogously. □

Corollary 2.11(ii) states the following: If μ is an eigenvalue of A in $\rho(B)$ then the function M_A has a pole at μ. In the next proposition we prove the same conclusion under a slightly different assumption: If μ is an eigenvalue of A of positive or of negative type and μ is no eigenvalue of the symmetric operator $S=A\cap B$, then M_A has a pole at μ (and, moreover, μ belongs to the resolvent set of B).

Proposition 2.16. *Let A, B, and I be as in Assumption* I, *let $S = A \cap B$, and $\mu \in I$. Then the following holds.*

(i) *If $\mu \in \sigma_{\pm\pm}(A) \backslash \sigma_{\mathrm{p}}(S)$ then M_A has a pole of order one at μ and $\mu \in \rho(B)$.*

(ii) *If $\mu \in \sigma_{\pm\pm}(B) \backslash \sigma_{\mathrm{p}}(S)$ then M_B has a pole of order one at μ and $\mu \in \rho(A)$.*

Proof. We verify assertion (i). The adjoint S^+ of $S = A \cap B$ is a closed linear relation with one-dimensional multivalued part if $\operatorname{dom} S$ is not dense, or an operator otherwise, cf. Sections 2.1 and 2.2. In both cases S^+ is a one-dimensional extension of A and B, and in both cases we regard S^+ as a linear relation and denote the elements in S^+ in the form $\{f, f'\}$. Let λ_0 be as in (2.5) and let $\varphi_A \in \mathcal{K}$ be as in Proposition 2.10(i). By Proposition 2.10(iii) we have for $y \in \mathcal{K}$

$$(A - \overline{\lambda}_0)^{-1} y - (B - \overline{\lambda}_0)^{-1} y = \frac{1}{M_A(\overline{\lambda}_0)} [y, \varphi_A] \gamma_A(\overline{\lambda}_0)$$

and the left hand side (and, hence the right hand side) is zero if and only if $y \in \operatorname{ran}(S - \overline{\lambda}_0)$. Thus $\varphi_A \in (\operatorname{ran}(S - \overline{\lambda}_0))^{[\perp]} = \ker(S^+ - \lambda_0)$ and we have the direct sum decomposition

$$S^+ = A \dotplus \big\{ \alpha \{\varphi_A, \lambda_0 \varphi_A\} \,|\, \alpha \in \mathbb{C} \big\},$$

cf. Proposition 2.4. Accordingly, we write $\{f, f'\} = \{f_A + \alpha\varphi_A, A f_A + \alpha\lambda_0\varphi_A\} \in S^+$ for some $f_A \in \operatorname{dom} A$. Suppose now that μ is an eigenvalue of positive or negative type of A such that $\mu \notin \sigma_{\mathrm{p}}(S)$, let $g_\mu \in \ker(A - \mu)$ be non-zero and denote the selfadjoint projection in $(\mathcal{K}, [\cdot,\cdot])$ onto the Hilbert (or anti-Hilbert) space $(\ker(A - \mu), [\cdot,\cdot])$ by P_μ. Since A is selfadjoint we obtain

$$\begin{aligned} [f', g_\mu] - [f, A g_\mu] &= [A f_A + \alpha\lambda_0\varphi_A, g_\mu] - [f_A + \alpha\varphi_A, A g_\mu] \\ &= [\alpha\lambda_0\varphi_A, g_\mu] - [\alpha\varphi_A, \mu g_\mu] = \alpha(\lambda_0 - \mu)[P_\mu\varphi_A, g_\mu]. \end{aligned}$$

Hence

$$P_\mu \varphi_A \neq 0 \tag{2.14}$$

as otherwise $\{g_\mu, A g_\mu\} \in S^{++} = S$ and $g_\mu \in \operatorname{dom} S$ with $S g_\mu = \mu g_\mu$, which is impossible as $\mu \notin \sigma_{\mathrm{p}}(S)$. On the other hand, for $\lambda \in \rho(A)$ we have by Proposition 2.10(i) $(A - \lambda)^{-1}\varphi_A = \frac{1}{\lambda - \lambda_0}(\gamma_A(\lambda) - \varphi_A)$ and $\gamma_A(\lambda_0) = \varphi_A$. Hence,

$$\begin{aligned} [(A - \lambda)^{-1}\varphi_A, \varphi_A] &= \frac{[\gamma_A(\lambda), \gamma_A(\lambda_0)] - [\varphi_A, \varphi_A]}{\lambda - \lambda_0} \\ &= \frac{M_A(\lambda)}{(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)} - \frac{M_A(\overline{\lambda}_0)}{(\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)} - \frac{[\varphi_A, \varphi_A]}{\lambda - \lambda_0}, \end{aligned} \tag{2.15}$$

cf. for example [41, Proof of Theorem 1.1]. Thus, if the function M_A admits an analytic continuation into the point μ, by the above formula (2.15) also the function $\lambda \mapsto [(A -$

$\lambda)^{-1}\varphi_A, \varphi_A]$ admits an analytic continuation into μ and

$$[P_\mu \varphi_A, \varphi_A] = -\frac{1}{2\pi i}\int_{\mathcal{C}_\mu} [(A-\lambda)^{-1}\varphi_A, \varphi_A]\, d\lambda = 0,$$

where the above contour integral is along a sufficiently small circle $\mathcal{C}_\mu$ containing μ. As $(\ker(A-\mu), [\cdot,\cdot])$ is a Hilbert (or anti-Hilbert) space this implies $P_\mu\varphi_A = 0$, a contradiction to (2.14). Thus M_A can not be continued analytically into μ. By Assumption I μ is a pole of the resolvent of A and as $\mu \in \sigma_{\pm\pm}(A)$, this pole is of order one, cf. Section 1.2. By (2.15) M_A has as well a pole of order one at μ.

The same reasoning applies to the first assertion in (ii). Hence every eigenvalue of positive or negative type of B which is not an eigenvalue of S is a pole of first order of M_B.

In order to complete the proof of (i) we have to show $\mu \in \rho(B)$. As $\mu \notin \sigma_{\mathrm{p}}(S)$ the dimension of $\ker(B-\mu)$ is at most one. By the above reasoning M_A has a pole at μ, hence $M_B = -M_A^{-1}$ has a zero at μ. It then follows from the first assertion in (ii) that $\mu \notin \sigma_{\pm\pm}(B)$. Thus it remains to exclude the possibility of a neutral eigenvector of B corresponding to μ. In fact, if there is a neutral eigenvector there exists a Jordan chain of length greater one since $\mathfrak{L}_\mu(B)$ is a Krein space by Proposition 1.12. This results in a pole of at least second order of the resolvent of B at μ, see Proposition 1.11. But as $\mu \in \sigma_{\pm\pm}(A)$ the resolvent of A, γ_A and, as shown above, also M_A have poles of first order at μ. Therefore by Proposition 2.10(iii) the resolvent of B has a pole of at most first order at μ, a contradiction. We have shown $\mu \in \rho(B)$. □

2.4 Realisation of Weyl Functions

Given a function g, how to construct a boundary triplet such that g is the corresponding Weyl function? In this section we answer this question for rational functions g and provide explicit formulas. Similar constructions can be found in [16, Section 5.2], [12, Section 4.3], [22, Section 4], and [17, 23]. More abstract results on the existence of such realisations can for example be found in [14, 15, 34, 36, 67]; cf. also [60, 61, 68].

In regard of Proposition 2.6(iii), we can assume $\overline{g(\lambda)} = g(\overline{\lambda})$ for $\lambda \in \mathbb{C}$. We show the realisation of isolated poles and zeros in $\mathbb{R}$ and complex conjugate pairs of isolated poles and zeros in $\mathbb{C}\backslash\mathbb{R}$. Afterwards, we show how to realise translations and sums of Weyl functions. With this, the realisation of rational Weyl functions symmetric with respect to $\mathbb{R}$ is clear. Finally, we show that each rational function g which has no pole at ∞ and is symmetric with respect to $\mathbb{R}$ admits a realisation as a Weyl function with boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ such that $\ker\Gamma_0$ is a matrix.

For $n \in \mathbb{N}$ denote by

$$\operatorname{sip}(n) = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \tag{2.16}$$

the n-dimensional *sip-matrix* (cf. [51]). This matrix is unitary and selfadjoint in the Hilbert space $\mathbb{C}^n$ with the euclidean scalar product $\langle\cdot,\cdot\rangle$. Hence, $\mathbb{C}^n$ equipped with $[\cdot,\cdot] = \langle J\cdot,\cdot\rangle$, $J = \operatorname{sip}(n)$ or $J = -\operatorname{sip}(n)$, is a Krein space. In the following denote by $\{e_1,\dots,e_n\}$ the standard basis of $\mathbb{C}^n$. We start with the realisation of isolated poles and zeros of order one in $\mathbb{R}$.

Proposition 2.17. *For $c>0$, $\mu\in\mathbb{R}$, and $\sigma\in\{\pm1\}$, let*

$$g_{-1}(\lambda) = \frac{\sigma c}{(\lambda-\mu)}, \quad \lambda\in\mathbb{C}\backslash\{\mu\}, \qquad \textit{and} \qquad g_1(\lambda) = \sigma c(\lambda-\mu), \quad \lambda\in\mathbb{C}.$$

Let $\mathbb{C}$ be equipped with the inner products $[\cdot,\cdot]_1 := -\sigma\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]_2 := \sigma\langle\cdot,\cdot\rangle$. Define the relation $S=\{0,0\}$. Then $S^+ = \mathbb{C}^2$ with respect to $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ and the following statements hold.

(i) *Consider $(\mathbb{C},[\cdot,\cdot]_1)$. The triplet $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ given by*

$$\begin{aligned}\Gamma_0 : S^+\to\mathbb{C},\quad &\Gamma_0\big(\{f,f'\}\big) = c^{-1/2}(\mu f - f'),\\ \Gamma_1 : S^+\to\mathbb{C},\quad &\Gamma_1\big(\{f,f'\}\big) = -\sigma c^{1/2} f,\end{aligned}$$

is a boundary triplet for S^+ and the corresponding Weyl function M fulfils $M = g_{-1}$.

(ii) *Consider $(\mathbb{C},[\cdot,\cdot]_2)$. The triplet $\{\mathbb{C},\underline{\Gamma}_0,\underline{\Gamma}_1\}$ given by*

$$\begin{aligned}\underline{\Gamma}_0 : S^+\to\mathbb{C},\quad &\underline{\Gamma}_0\big(\{f,f'\}\big) = -\sigma c^{-1/2} f,\\ \underline{\Gamma}_1 : S^+\to\mathbb{C},\quad &\underline{\Gamma}_1\big(\{f,f'\}\big) = c^{1/2}(\mu f - f'),\end{aligned}$$

is a boundary triplet for S^+ and the corresponding Weyl function $\underline{M}$ fulfils $\underline{M} = g_1$.

Proof. It is clear that $S^+=\mathbb{C}^2$ holds with respect to $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$.
(i) We show Green's identity (2.1). Let $\widehat{f} = \{f,f'\}, \widehat{g} = \{g,g'\}\in S^+$, then

$$\begin{aligned}[f',g]_1 - [f,g']_1 &= \sigma\big(\langle f,g'\rangle - \langle f',g\rangle\big)\\ &= \langle\sigma f,g'\rangle - \langle\sigma f,\mu g\rangle - \langle f',\sigma g\rangle + \langle\mu f,\sigma g\rangle\\ &= \langle\sigma c^{1/2} f, c^{-1/2}(g'-\mu g)\rangle - \langle c^{-1/2}(f'-\mu f),\sigma c^{1/2} g\rangle\\ &= \langle\Gamma_1\widehat{f},\Gamma_0\widehat{g}\rangle - \langle\Gamma_0\widehat{f},\Gamma_1\widehat{g}\rangle.\end{aligned}$$

Obviously, $\Gamma = (\Gamma_0,\Gamma_1)^\top$ is surjective and hence $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ is a boundary triplet for S^+. Set $A_0 := \ker\Gamma_0 = \{\{f,\mu f\} | f\in\mathbb{C}\}$. As $S^+=\mathbb{C}^2$, the defect relation of S^+ at $\lambda\neq\mu$ is $\widehat{N}_\lambda(S^+) = \operatorname{span}\{\{1,\lambda\}\}$. Thus for $\lambda\in\rho(A_0) = \mathbb{C}\backslash\{\mu\}$ the Weyl function M fulfils by Proposition 2.6(ii)

$$-\sigma c^{1/2} f_\lambda = \Gamma_1\widehat{f}_\lambda = M(\lambda)\Gamma_0\widehat{f}_\lambda = M(\lambda)c^{-1/2}(\mu f_\lambda - \lambda f_\lambda) = M(\lambda)c^{-1/2}(\mu-\lambda)f_\lambda,$$

for $\widehat{f}_\lambda = (f_\lambda,\lambda f_\lambda)^\top\in\widehat{N}_\lambda(S^+)$. Thus, $M(\lambda) = \frac{\sigma c}{\lambda-\mu}$, $\lambda\in\mathbb{C}\backslash\{\mu\}$, which shows (i).

To prove (ii) note that $[\cdot,\cdot]_2 = \sigma\langle\cdot,\cdot\rangle = -[\cdot,\cdot]_1$. For $\widehat{f} = \{f, f'\} \in S^+$ we see

$$\Gamma_1 \widehat{f} = c\underline{\Gamma}_0 \widehat{f} \quad \text{and} \quad \Gamma_0 \widehat{f} = c^{-1}\underline{\Gamma}_1 \widehat{f}.$$

Hence, for $\widehat{g} = \{g, g'\} \in S^+$ it follows with (i)

$$\begin{aligned}
[f', g]_2 - [f, g']_2 &= \langle f', J_2 g\rangle - \langle J_2 f, g'\rangle = \langle J_1 f, g'\rangle - \langle f', J_1 g\rangle = \langle \Gamma_0 \widehat{f}, \Gamma_1 \widehat{g}\rangle - \langle \Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}\rangle \\
&= \langle c^{-1}\underline{\Gamma}_1 \widehat{f}, c\underline{\Gamma}_0 \widehat{g}\rangle - \langle c\underline{\Gamma}_0 \widehat{f}, c^{-1}\underline{\Gamma}_1 \widehat{g}\rangle = \langle \underline{\Gamma}_1 \widehat{f}, \underline{\Gamma}_0 \widehat{g}\rangle - \langle \underline{\Gamma}_0 \widehat{f}, \underline{\Gamma}_1 \widehat{g}\rangle,
\end{aligned}$$

which shows Green's equality. Moreover, $\underline{\Gamma} = (\underline{\Gamma}_0, \underline{\Gamma}_1)^\top$ is surjective and hence $\{\mathbb{C}, \underline{\Gamma}_0, \underline{\Gamma}_1\}$ is a boundary triplet for S^+ (with respect to $[\cdot,\cdot]_2$). Since the defect relation $\widehat{N}_\lambda(S^+)$ is independent of the chosen boundary triplet, we obtain with (i) for $\widehat{f}_\lambda \in \widehat{N}_\lambda(S^+)$

$$-c^{1/2}(\lambda - \mu) f_\lambda = c\Gamma_0 \widehat{f}_\lambda = \underline{\Gamma}_1 \widehat{f}_\lambda = \underline{M}(\lambda)\underline{\Gamma}_0 \widehat{f}_\lambda = \underline{M}(\lambda) c^{-1}\Gamma_1 \widehat{f}_\lambda = -\sigma c^{-1/2}\underline{M}(\lambda) f_\lambda,$$

for $\lambda \in \rho(\ker \underline{\Gamma}_0)$. This yields $\underline{M}(\lambda) = \sigma(\lambda - \mu)$, $\lambda \in \rho(\ker \underline{\Gamma}_0)$, which shows (ii). □

Remark 2.18. Note that the relation $\ker \underline{\Gamma}_0$ has a non-trivial multivalued part and therefore has no representation as operator (in contrast to $\ker \Gamma_0 = A_0$). Indeed, $\widehat{f} = \{f, f'\} \in S^+ = \mathbb{C}^2$ lies in the kernel of $\underline{\Gamma}_0$ if and only if $-\sigma c^{-1/2} f = 0$. Hence,

$$\ker \underline{\Gamma}_0 = \{0\} \times \mathbb{C}.$$

The next proposition shows how to realise isolated poles and zeros in $\mathbb{R}$ of higher order.

Proposition 2.19. *For some $n \geq 2$, $c > 0$, $\mu \in \mathbb{R}$, and $\sigma \in \{\pm 1\}$, let*

$$g_{-n}(\lambda) = \frac{\sigma c}{(\lambda - \mu)^n}, \quad \lambda \in \mathbb{C}\backslash\{\mu\}, \quad \text{and} \quad g_n(\lambda) = \sigma c(\lambda - \mu)^n, \quad \lambda \in \mathbb{C}.$$

Let A_0 be the $(n \times n)$-matrix

$$A_0 := \begin{pmatrix} \mu & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \mu \end{pmatrix},$$

and let S be the restriction of A_0 to $\{e_2, \dots, e_n\}$,

$$S := \big\{\{f, A_0 f\} \in A_0 \,\big|\, f \in \operatorname{span}\{e_2, \dots, e_n\}\big\}.$$

Define the fundamental symmetries $J_1 := -\sigma \operatorname{sip}(n)$ and $J_2 := \sigma \operatorname{sip}(n)$ and the inner products $[\cdot,\cdot]_1 := \langle J\cdot,\cdot\rangle_1$ and $[\cdot,\cdot]_2 := \langle J\cdot,\cdot\rangle_2$. Then the adjoint S^+ of S with respect to

$[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ is given by

$$S^+ = \Big\{\{f,f'\} \in \mathbb{C}^n \times \mathbb{C}^n \,\Big|\, f = \sum_{j=1}^{n} \alpha_j e_j,\ f' = \alpha_n' e_n + \sum_{j=1}^{n-1} (\mu\alpha_j + \alpha_{j+1}) e_j,\ \alpha_1, \ldots, \alpha_n, \alpha_n' \in \mathbb{C}\Big\}. \tag{2.17}$$

Moreover, the following statements hold.

(i) *Consider $(\mathbb{C}^n, [\cdot,\cdot]_1)$. The triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ given by*

$$\begin{aligned} \Gamma_0 &: S^+ \to \mathbb{C}, \quad \Gamma_0\big(\{f,f'\}\big) = c^{-1/2}\left(\mu\langle f, e_n\rangle - \langle f', e_n\rangle\right), \\ \Gamma_1 &: S^+ \to \mathbb{C}, \quad \Gamma_1\big(\{f,f'\}\big) = -\sigma c^{1/2}\langle f, e_1\rangle, \end{aligned}$$

is a boundary triplet for S^+ and the corresponding Weyl function M fulfils $M = g_{-n}$.

(ii) *Consider $(\mathbb{C}^n, [\cdot,\cdot]_2)$. The triplet $\{\mathbb{C}, \underline{\Gamma}_0, \underline{\Gamma}_1\}$ given by*

$$\begin{aligned} \underline{\Gamma}_0 &: S^+ \to \mathbb{C}, \quad \underline{\Gamma}_0\big(\{f,f'\}\big) = -\sigma c^{-1/2}\langle f, e_1\rangle, \\ \underline{\Gamma}_1 &: S^+ \to \mathbb{C}, \quad \underline{\Gamma}_1\big(\{f,f'\}\big) = c^{1/2}\left(\mu\langle f, e_n\rangle - \langle f', e_n\rangle\right), \end{aligned}$$

is a boundary triplet for S^+ and the corresponding Weyl function $\underline{M}$ fulfils $\underline{M} = g_n$.

Proof. First we show that S^+ has the form (2.17) with respect to $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$. For $\{g, g'\} \in S$,

$$g = \sum_{j=2}^{n} \beta_j e_j, \quad g' = A_0 g = \mu \sum_{j=2}^{n} \beta_j e_j + \sum_{j=1}^{n-1} \beta_{j+1} e_j,$$

with $\beta_2, \ldots, \beta_n \in \mathbb{C}$, and $i = 1, 2$, we have

$$J_i g = (-1)^i \sigma \sum_{j=1}^{n-1} \beta_{n+1-j} e_j, \quad J_i g' = (-1)^i \sigma \Big(\mu \sum_{j=1}^{n-1} \beta_{n+1-j} e_j + \sum_{j=2}^{n} \beta_{n+2-j} e_j\Big).$$

Therefore, for $\{f, f'\} \in S^+$ with $f = \sum_{j=1}^n \alpha_j e_j$, $f' = \sum_{j=1}^n \alpha_j' e_j$, $\alpha_j, \alpha_j' \in \mathbb{C}$, $j = 1, \ldots, n$, we obtain for $i = 1, 2$

$$\begin{aligned} 0 = [f', g] - [f, g'] &= \langle f', J_i g\rangle - \langle f, J_i g'\rangle \\ &= \Big\langle \sum_{j=1}^{n} \alpha_j' e_j, (-1)^i \sigma \sum_{j=1}^{n-1} \beta_{n+1-j} e_j \Big\rangle - \Big\langle \sum_{j=1}^{n} \alpha_j e_j, (-1)^i \sigma \Big(\mu \sum_{j=1}^{n-1} \beta_{n+1-j} e_j + \sum_{j=2}^{n} \beta_{n+2-j} e_j\Big)\Big\rangle \\ &= (-1)^i \sigma \Big(\sum_{j=1}^{n-1} \alpha_j' \overline{\beta_{n+1-j}} - \mu \sum_{j=1}^{n-1} \alpha_j \overline{\beta_{n+1-j}} - \sum_{j=2}^{n} \alpha_j \overline{\beta_{n+2-j}}\Big) \\ &= (-1)^i \sigma \sum_{j=1}^{n-1} \left(\alpha_j' - \mu\alpha_j - \alpha_{j+1}\right) \overline{\beta_{n+1-j}}. \end{aligned}$$

Hence, $\alpha_j' = \mu\alpha_j + \alpha_{j+1}$, $j = 1, \ldots, n-1$, for $i = 1, 2$, which shows (2.17).

We show (i). Let Γ_0, Γ_1 be defined as in (i). Obviously, $\Gamma = (\Gamma_0, \Gamma_1)^\top$ is surjective. Moreover, $A_0 = \ker \Gamma_0$. Indeed, let $\{f, f'\} \in S^+$ as in (2.17). Then $\{f, f'\} \in \ker \Gamma_0$ if and only if $\alpha_n' = \mu\alpha_n$, i.e. $f' = A_0 f$.

We show the abstract Green identity (2.1). For this let $\{f, f'\}, \{g, g'\} \in S^+$,

$$
\begin{aligned}
f &= \sum_{j=1}^{n} \alpha_j e_j, \quad f' = \alpha_n' e_n + \sum_{j=1}^{n-1} (\mu\alpha_j + \alpha_{j+1})\, e_j,\\
g &= \sum_{j=1}^{n} \beta_j e_j, \quad g' = \beta_n' e_n + \sum_{j=1}^{n-1} (\mu\beta_j + \beta_{j+1})\, e_j.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&[f', g]_1 - [f, g']_1 = \langle f', J_1 g\rangle - \langle J_1 f, g'\rangle \\
&= \Big\langle \alpha_n' e_n + \sum_{j=1}^{n-1} (\mu\alpha_j + \alpha_{j+1})\, e_j, -\sigma \sum_{j=1}^{n} \beta_{n+1-j} e_j \Big\rangle \\
&\qquad - \Big\langle -\sigma \sum_{j=1}^{n} \alpha_{n+1-j} e_j,\ \beta_n' e_n + \sum_{j=1}^{n-1} (\mu\beta_j + \beta_{j+1})\, e_j \Big\rangle \\
&= -\sigma\Big(\alpha_n' \overline{\beta_1} + \sum_{j=1}^{n-1} (\mu\alpha_j + \alpha_{j+1})\, \overline{\beta_{n+1-j}} - \alpha_1 \overline{\beta_n'} - \sum_{j=1}^{n-1} \alpha_{n+1-j} \left(\mu\overline{\beta_j} + \overline{\beta_{j+1}} \right) \Big) \\
&= -\sigma\Big(\alpha_n' \overline{\beta_1} + \mu \sum_{j=1}^{n-1} \alpha_j \overline{\beta_{n+1-j}} + \sum_{j=1}^{n-1} \alpha_{j+1} \overline{\beta_{n+1-j}} - \alpha_1 \overline{\beta_n'} - \mu \sum_{j=2}^{n} \alpha_j \overline{\beta_{n+1-j}} - \sum_{j=1}^{n-1} \alpha_{j+1} \overline{\beta_{n+1-j}} \Big) \\
&= -\sigma \left(\mu \left(\alpha_1 \overline{\beta_n} - \alpha_n \overline{\beta_1} \right) + \alpha_n' \overline{\beta_1} - \alpha_1 \overline{\beta_n'} \right) \\
&= -\sigma\alpha_1 (\mu\overline{\beta_n} - \overline{\beta_n'}) - (\mu\alpha_n - \alpha_n')(-\sigma\overline{\beta_1}) \\
&= -\sigma c^{1/2} \alpha_1 c^{-1/2} \left(\mu\overline{\beta_n} - \overline{\beta_n'} \right) - c^{-1/2} \left(\mu\alpha_n - \alpha_n' \right) (-\sigma c^{1/2} \overline{\beta_1}) \\
&= \left\langle \Gamma_1 \widehat{f}, \Gamma_0 \widehat{g} \right\rangle - \left\langle \Gamma_0 \widehat{f}, \Gamma_1 \widehat{g} \right\rangle.
\end{aligned}
$$

Hence, $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^+. We calculate the defect relation. For $\widehat{f}_\lambda = \{f_\lambda, f_\lambda'\} \in \widehat{N}_\lambda(S^+) \subseteq S^+$, $\lambda \neq \mu$, we have $f_\lambda' = \lambda f_\lambda$ and hence

$$
\begin{aligned}
\mu\alpha_j + \alpha_{j+1} &= \lambda\alpha_j, \quad j = 1, \ldots, n-1, \quad \text{and}\\
\alpha_n' &= \lambda\alpha_n.
\end{aligned}
$$

This yields

$$
\widehat{N}_\lambda(S^+) = \left\{ \widehat{f}_\lambda = \{\alpha f_\lambda, \alpha\lambda f_\lambda\} \,\middle|\, f_\lambda = \left(1, (\lambda-\mu), \ldots, (\lambda-\mu)^{n-1}\right)^\top, \alpha \in \mathbb{C} \right\}
$$

and we see

$$\Gamma_0\widehat{f}_\lambda = \alpha c^{-1/2}(\mu(\lambda-\mu)^{n-1} - \lambda(\lambda-\mu)^{n-1}) = -\alpha c^{-1/2}(\lambda-\mu)^n,$$
$$\Gamma_1\widehat{f}_\lambda = -\alpha\sigma c^{1/2}.$$

By Proposition 2.6(ii) the corresponding Weyl function M satisfies

$$-\alpha\sigma c^{1/2} = \Gamma_1\widehat{f}_\lambda = M(\lambda)\Gamma_0\widehat{f}_\lambda = -\alpha M(\lambda)c^{-1/2}(\lambda-\mu)^n,$$

for $\widehat{f}_\lambda \in \widehat{N}_\lambda(S^+)$, $\lambda \in \rho(A_0) = \mathbb{C}\backslash\{\mu\}$, i.e. $M(\lambda) = \frac{\sigma c}{(\lambda-\mu)^n} = g(\lambda)$, $\lambda \in \mathbb{C}\backslash\{\mu\}$, which concludes the proof of (i).

The proof of (ii) is analogous to the proof of Proposition 2.17(ii). Note that $[\cdot,\cdot]_2 = -[\cdot,\cdot]_1$. For $\widehat{f} = \{f, f'\} \in S^+$ we see

$$\Gamma_1\widehat{f} = c\underline{\Gamma}_0\widehat{f} \quad \text{and} \quad \Gamma_0\widehat{f} = c^{-1}\underline{\Gamma}_1\widehat{f}.$$

Hence, for $\widehat{g} = \{g, g'\} \in S^+$ it follows with (i)

$$\begin{aligned}[f', g]_2 - [f, g']_2 &= \langle f', J_2 g\rangle - \langle J_2 f, g'\rangle = \langle J_1 f, g'\rangle - \langle f', J_1 g\rangle = \langle \Gamma_0\widehat{f}, \Gamma_1\widehat{g}\rangle - \langle \Gamma_1\widehat{f}, \Gamma_0\widehat{g}\rangle \\ &= \langle c^{-1}\underline{\Gamma}_1\widehat{f}, c\underline{\Gamma}_0\widehat{g}\rangle - \langle c\underline{\Gamma}_0\widehat{f}, c^{-1}\underline{\Gamma}_1\widehat{g}\rangle = \langle \underline{\Gamma}_1\widehat{f}, \underline{\Gamma}_0\widehat{g}\rangle - \langle \underline{\Gamma}_0\widehat{f}, \underline{\Gamma}_1\widehat{g}\rangle,\end{aligned}$$

which shows the abstract Green equality. Moreover, $\underline{\Gamma} = (\underline{\Gamma}_0, \underline{\Gamma}_1)^\top$ is surjective and hence $\{\mathbb{C}, \underline{\Gamma}_0, \underline{\Gamma}_1\}$ is a boundary triplet for S^+ (with respect to $[\cdot,\cdot]_2$). We obtain with (i) for $\widehat{f}_\lambda = \{\alpha f_\lambda, \alpha\lambda f_\lambda\} \in \widehat{N}_\lambda(S^+)$

$$-\alpha c^{1/2}(\lambda-\mu)^n = c\Gamma_0\widehat{f}_\lambda = \underline{\Gamma}_1\widehat{f}_\lambda = \underline{M}(\lambda)\underline{\Gamma}_0\widehat{f}_\lambda = \underline{M}(\lambda)c^{-1}\Gamma_1\widehat{f}_\lambda = -\alpha\sigma c^{-1/2}\underline{M}(\lambda)$$

for $\lambda \in \rho(\ker\underline{\Gamma}_0)$. This yields $\underline{M}(\lambda) = \sigma(\lambda-\mu)^n$, $\lambda \in \rho(\ker\underline{\Gamma}_0)$, which shows (ii). □

Remark 2.20. Analogously to Remark 2.18 we see that the relation $\ker\underline{\Gamma}_0$ has a non-trivial multivalued part and therefore has no representation as operator (in contrast to $\ker\Gamma_0 = A_0$).

Since we assume $\overline{g(\lambda)} = g(\overline{\lambda})$, $\lambda \in \mathbb{C}$, non-real poles and zeros of g always come in complex conjugate pairs. The realisation of such a pair is treated in the next proposition.

Proposition 2.21. *For some $n \geq 1$, $c > 0$, $\mu \in \mathbb{C}\backslash\mathbb{R}$, and $\sigma \in \{\pm 1\}$, let*

$$g_{-n}(\lambda) = \frac{\sigma c}{(\lambda-\mu)^n} + \frac{\sigma c}{(\lambda-\overline{\mu})^n},\ \lambda \in \mathbb{C}\backslash\{\mu, \overline{\mu}\},\ \text{and } g_n(\lambda) = \sigma c(\lambda-\mu)^n + \sigma c(\lambda-\overline{\mu})^n,\ \lambda \in \mathbb{C}.$$

Let A_0 be the $(2n \times 2n)$-matrix

$$A_0 := \begin{pmatrix} \mu & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & 1 & & & & \\ & & & \mu & 0 & & & \\ & & & & \overline{\mu} & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & 1 \\ & & & & & & & \overline{\mu} \end{pmatrix},$$

and let the restriction S of A_0 be given by

$$S := \big\{ \{f, A_0 f\} \in A_0 \,\big|\, f \in \operatorname{span}\{e_1 - e_{n+1}, e_2, \dots, e_n, e_{n+2}, \dots, e_{2n}\} \big\} \subseteq \mathbb{C}^{2n} \times \mathbb{C}^{2n}.$$

Define the fundamental symmetries $J_1 := -\sigma \operatorname{sip}(2n)$ and $J_2 := \sigma \operatorname{sip}(2n)$ and the inner products $[\cdot,\cdot]_1 := \langle J\cdot,\cdot\rangle_1$ and $[\cdot,\cdot]_2 := \langle J\cdot,\cdot\rangle_2$. Then the adjoint S^+ of S with respect to $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ is given by

$$\begin{aligned} S^+ = \Bigg\{ \{f, f'\} \in \mathbb{C}^{2n} \times \mathbb{C}^{2n} \,\Bigg|\, & \alpha_1, \dots, \alpha_{2n}, \alpha_n', \alpha_{2n}' \in \mathbb{C},\ \mu\alpha_n - \alpha_n' = \overline{\mu}\alpha_{2n} - \alpha_{2n}', \\ & f = \sum_{j=1}^{2n} \alpha_j e_j,\ f' = \alpha_n' e_n + \sum_{j=1}^{n-1} (\mu\alpha_j + \alpha_{j+1}) e_j + \alpha_{2n}' e_{2n} + \sum_{j=n+1}^{2n-1} (\overline{\mu}\alpha_j + \alpha_{j+1}) e_j \Bigg\}. \end{aligned} \tag{2.18}$$

Moreover, the following statements hold.

(i) *Consider $(\mathbb{C}^{2n}, [\cdot,\cdot]_1)$. The triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ given by*

$$\begin{aligned} \Gamma_0 &: S^+ \to \mathbb{C}, \quad \Gamma_0\big(\{f, f'\}\big) = c^{-1/2}\left(\mu\langle f, e_n\rangle - \langle f', e_n\rangle\right), \\ \Gamma_1 &: S^+ \to \mathbb{C}, \quad \Gamma_1\big(\{f, f'\}\big) = -\sigma c^{1/2}\left(\langle f, e_1\rangle + \langle f, e_{n+1}\rangle\right), \end{aligned}$$

is a boundary triplet for S^+ and the corresponding Weyl function M fulfils $M = g_{-n}$.

(ii) *Consider $(\mathbb{C}^{2n}, [\cdot,\cdot]_2)$. The triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ given by*

$$\begin{aligned} \underline{\Gamma}_0 &: S^+ \to \mathbb{C}, \quad \underline{\Gamma}_0\big(\{f, f'\}\big) = -\sigma c^{-1/2}\left(\langle f, e_1\rangle + \langle f, e_{n+1}\rangle\right), \\ \underline{\Gamma}_1 &: S^+ \to \mathbb{C}, \quad \underline{\Gamma}_1\big(\{f, f'\}\big) = c^{1/2}\left(\mu\langle f, e_n\rangle - \langle f', e_n\rangle\right), \end{aligned}$$

is a boundary triplet for S^+ and the corresponding Weyl function $\underline{M}$ fulfils $\underline{M} = g_n$.

Proof. As in the proof of Proposition 2.19, we show that S^+ has the form (2.18) with respect to $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$. For $\{g,g'\}\in S$,

$$g=\beta_1(e_1-e_{n+1})+\sum_{j=2}^{n}\beta_j e_j+\sum_{j=n+2}^{2n}\beta_j e_j,$$
$$g'=A_0 g=\beta_1(\mu e_1-\overline{\mu}e_{n+1})+\mu\sum_{j=2}^{n}\beta_j e_j+\sum_{j=1}^{n-1}\beta_{j+1}e_j+\overline{\mu}\sum_{j=n+2}^{2n}\beta_j e_j+\sum_{j=n+1}^{2n-1}\beta_{j+1}e_j,$$

with $\beta_1,\dots,\beta_n,\beta_{n+2},\dots,\beta_{2n}\in\mathbb{C}$, we have

$$\begin{aligned}
J_i g&=(-1)^i\sigma\Big(\beta_1(e_{2n}-e_n)+\sum_{j=1}^{n-1}\beta_{2n+1-j}e_j+\sum_{j=n+1}^{2n-1}\beta_{2n+1-j}e_j\Big),\\
J_i g'&=(-1)^i\sigma\Big(\beta_1(\mu e_{2n}-\overline{\mu}e_n)+\overline{\mu}\sum_{j=1}^{n-1}\beta_{2n+1-j}e_j+\sum_{j=2}^{n}\beta_{2n+2-j}e_j\\
&\qquad+\mu\sum_{j=n+1}^{2n-1}\beta_{2n+1-j}e_j+\sum_{j=n+2}^{2n}\beta_{2n+2-j}e_j\Big),
\end{aligned}$$

$i=1,2$. Let $\{f,f'\}\in S^+$ with $f=\sum_{j=1}^{2n}\alpha_j e_j$, $f'=\sum_{j=1}^{2n}\alpha'_j e_j$, $\alpha_j,\alpha'_j\in\mathbb{C}$, $j=1,\dots,2n$. For $i=1,2$ we see

$$\begin{aligned}
0&=[f',g]_i-[f,g']_i=\langle f',J_i g\rangle-\langle f,J_i g'\rangle\\
&=\Big\langle\sum_{j=1}^{2n}\alpha'_j e_j,(-1)^i\sigma\Big(\beta_1(e_{2n}-e_n)+\sum_{j=1}^{n-1}\beta_{2n+1-j}e_j+\sum_{j=n+1}^{2n-1}\beta_{2n+1-j}e_j\Big)\Big\rangle\\
&\quad-\Big\langle\sum_{j=1}^{2n}\alpha_j e_j,(-1)^i\sigma\Big(\beta_1(\mu e_{2n}-\overline{\mu}e_n)+\overline{\mu}\sum_{j=1}^{n-1}\beta_{2n+1-j}e_j+\sum_{j=2}^{n}\beta_{2n+2-j}e_j\\
&\qquad+\mu\sum_{j=n+1}^{2n-1}\beta_{2n+1-j}e_j+\sum_{j=n+2}^{2n}\beta_{2n+2-j}e_j\Big)\Big\rangle\\
&=(-1)^i\sigma\Big(\sum_{j=1}^{n-1}\alpha'_j\overline{\beta_{n+1-j}}-\mu\sum_{j=1}^{n-1}\alpha_j\overline{\beta_{n+1-j}}-\sum_{j=2}^{n}\alpha_j\overline{\beta_{n+2-j}}\Big)\\
&=(-1)^i\sigma\sum_{j=1}^{n-1}\left(\alpha'_j-\mu\alpha_j-\alpha_{j+1}\right)\overline{\beta_{n+1-j}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu\alpha_n-\alpha'_n&=\overline{\mu}\alpha_{2n}-\alpha'_{2n},\\
\alpha'_j&=\mu\alpha_j+\alpha_{j+1},\quad j=1,\dots,n-1,\\
\alpha'_j&=\overline{\mu}\alpha_j+\alpha_{j+1},\quad j=n+1,\dots,2n-1,
\end{aligned}$$

for $i=1,2$, which shows (2.18).

(i) Let Γ_0, Γ_1 be defined as in (i). Then $\Gamma = (\Gamma_0, \Gamma_1)^\top$ is surjective and $A_0 = \ker \Gamma_0$. Indeed, let $\{f, f'\} \in S^+$ as in (2.18). Then $\{f, f'\} \in \ker \Gamma_0$ if and only if $\alpha'_n = \mu\alpha_n$ (and thus $\alpha'_{2n} = \overline{\mu}\alpha_{2n}$), i.e. $f' = A_0 f$.

To show Green's identity (2.1) let $\{f, f'\}, \{g, g'\} \in S^+$,

$$
\begin{aligned}
f &= \sum_{j=1}^{2n} \alpha_j e_j, \quad f' = \alpha'_n e_n + \sum_{j=1}^{n-1} (\mu\alpha_j + \alpha_{j+1}) e_j + \alpha'_{2n} e_{2n} + \sum_{j=n+1}^{2n-1} (\overline{\mu}\alpha_j + \alpha_{j+1}) e_j, \\
&\qquad \mu\alpha_n - \alpha'_n = \overline{\mu}\alpha_{2n} - \alpha'_{2n}, \\
g &= \sum_{j=1}^{2n} \beta_j e_j, \quad g' = \beta'_n e_n + \sum_{j=1}^{n-1} (\mu\beta_j + \beta_{j+1}) e_j + \beta'_{2n} e_{2n} + \sum_{j=n+1}^{2n-1} (\overline{\mu}\beta_j + \beta_{j+1}) e_j, \\
&\qquad \mu\beta_n - \beta'_n = \overline{\mu}\beta_{2n} - \beta'_{2n}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&[f', g]_1 - [f, g']_1 = \big\langle f', J_1 g \big\rangle - \big\langle J_1 f, g' \big\rangle \\
&= \Big\langle \alpha'_n e_n + \sum_{j=1}^{n-1} (\mu\alpha_j + \alpha_{j+1}) e_j + \alpha'_{2n} e_{2n} + \sum_{j=n+1}^{2n-1} (\overline{\mu}\alpha_j + \alpha_{j+1}) e_j,\, -\sigma \sum_{j=1}^{2n} \beta_{2n+1-j} e_j \Big\rangle \\
&\qquad - \Big\langle -\sigma \sum_{j=1}^{2n} \alpha_{2n+1-j} e_j,\, \beta'_n e_n + \sum_{j=1}^{n-1} (\mu\beta_j + \beta_{j+1}) e_j + \beta'_{2n} e_{2n} + \sum_{j=n+1}^{2n-1} (\overline{\mu}\beta_j + \beta_{j+1}) e_j \Big\rangle \\
&= -\sigma \Big(\alpha'_n \overline{\beta_{n+1}} + \sum_{j=1}^{n-1} (\mu\alpha_j + \alpha_{j+1}) \overline{\beta_{2n+1-j}} + \alpha'_{2n} \overline{\beta_1} + \sum_{j=n+1}^{2n-1} (\overline{\mu}\alpha_j + \alpha_{j+1}) \overline{\beta_{2n+1-j}} \\
&\qquad - \alpha_{n+1} \overline{\beta'_n} - \sum_{j=1}^{n-1} \alpha_{2n+1-j} \big(\overline{\mu} \overline{\beta_j} + \overline{\beta_{j+1}} \big) - \alpha_1 \overline{\beta'_{2n}} - \sum_{j=n+1}^{2n-1} \alpha_{2n+1-j} \big(\mu \overline{\beta_j} + \overline{\beta_{j+1}} \big) \Big) \\
&= -\sigma \left(\alpha'_n \overline{\beta_{n+1}} + \alpha'_{2n} \overline{\beta_1} - \alpha_{n+1} \overline{\beta'_n} - \alpha_1 \overline{\beta'_{2n}} + \mu \left(\alpha_1 \overline{\beta_{2n}} - \alpha_n \overline{\beta_{n+1}} \right) + \overline{\mu} \left(\alpha_{n+1} \overline{\beta_n} - \alpha_{2n} \overline{\beta_1} \right) \right) \\
&= -\sigma \Big((\alpha'_n - \mu\alpha_n) \overline{\beta_{n+1}} + (\alpha'_{2n} - \overline{\mu}\alpha_{2n}) \overline{\beta_1} \Big) - \sigma \Big(\alpha_{n+1} (\overline{\mu} \overline{\beta_n} - \overline{\beta'_n}) + \alpha_1 (\mu \overline{\beta_{2n}} - \overline{\beta'_{2n}}) \Big) \\
&= \sigma (\mu\alpha_n - \alpha'_n) (\overline{\beta_1} - \overline{\beta_{n+1}}) - \sigma (\alpha_1 + \alpha_{n+1}) (\overline{\mu} \overline{\beta_n} - \overline{\beta'_n}) \\
&= -\langle \Gamma_0 \widehat{f}, \Gamma_1 \widehat{g} \rangle + \langle \Gamma_1 \widehat{f}, \Gamma_0 \widehat{g} \rangle.
\end{aligned}
$$

Thus, $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^+. To determine the defect relation of S^+ at $\lambda \in \mathbb{C} \backslash \{\mu, \overline{\mu}\}$, let $\widehat{f}_\lambda = \{f_\lambda, f'_\lambda\} \in \widehat{N}_\lambda(S^+)$,

$$
f = \sum_{j=1}^{2n} \beta_j e_j, \quad f' = \sum_{j=1}^{n-1} (\mu\beta_j + \beta_{j+1}) e_j + \beta'_n e_n + \sum_{j=n+1}^{2n-1} (\overline{\mu}\beta_j + \beta_{j+1}) e_j + \beta'_{2n} e_{2n}
$$

with $\mu\beta_n - \beta'_n = \overline{\mu}\beta_{2n} - \beta'_{2n}$. As $f'_\lambda = \lambda f_\lambda$ holds, we have

$$
\lambda\beta_j = \mu\beta_j + \beta_{j+1}, \quad j = 1, \dots, n-1,
$$

$$\begin{aligned}\lambda\beta_n &= \beta_n',\\ \lambda\beta_j &= \overline{\mu}\beta_j + \beta_{j+1}, \quad j = n+1,\dots,2n-1,\\ \lambda\beta_{2n} &= \beta_{2n}',\end{aligned}$$

and therefore

$$\begin{aligned}\beta_n &= (\lambda-\mu)^k\beta_{n-k}, \quad\ k = 1,\dots,n-1,\\ \beta_{2n} &= (\lambda-\overline{\mu})^k\beta_{2n-k}, \quad k = 1,\dots,n-1, \quad \text{ and}\\ (\lambda-\mu)\beta_n &= (\lambda-\overline{\mu})\beta_{2n}.\end{aligned}$$

Setting $\beta_1 := (\lambda-\overline{\mu})^n$ we finally see

$$\widehat{N}_\lambda(S^+) = \operatorname{span}\left\{ \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix} \,\middle|\, f_\lambda = \sum_{j=1}^{n}(\lambda-\mu)^{j-1}(\lambda-\overline{\mu})^n e_j + \sum_{j=n+1}^{2n}(\lambda-\overline{\mu})^{j-n-1}(\lambda-\mu)^n e_j \right\}.$$

Hence, for $\widehat{f}_\lambda = \{\alpha f_\lambda, \alpha\lambda f_\lambda\} \in \widehat{N}_\lambda(S^+)$ we have by Proposition 2.6(ii) for the Weyl function M

$$\begin{aligned}-\alpha\sigma c^{1/2}\left((\lambda-\mu)^n + (\lambda-\overline{\mu})^n\right) &= \Gamma_1\widehat{f}_\lambda = M(\lambda)\Gamma_0\widehat{f}_\lambda\\ &= \alpha M(\lambda)c^{-1/2}\big(\mu(\lambda-\mu)^{n-1}(\lambda-\overline{\mu})^n - \lambda(\lambda-\mu)^{n-1}(\lambda-\overline{\mu})^n\big)\\ &= -\alpha M(\lambda)c^{-1/2}(\lambda-\mu)^n(\lambda-\overline{\mu})^n\end{aligned}$$

for $\lambda \in \rho(A_0) = \mathbb{C}\backslash\{\mu,\overline{\mu}\}$, i.e. $M(\lambda) = \frac{\sigma c}{(\lambda-\mu)^n} + \frac{\sigma c}{(\lambda-\overline{\mu})^n}$, $\lambda \in \mathbb{C}\backslash\{\mu,\overline{\mu}\}$.
Now (ii) follows as in the proofs of Proposition 2.17 and 2.19. □

In the next proposition we show how to realise a translation of a given Weyl function. This result follows as in the Hilbert space case from more general transformations, see for example [33, Section 3.3] and [35, Corollary 3]. For the convenience of the reader we provide a short proof.

Proposition 2.22. *Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space with fundamental symmetry J. Consider a closed symmetric relation $S \subseteq \mathcal{K}^2$. Let $\{\mathbb{C}, \underline{\Gamma}_0, \underline{\Gamma}_1\}$ be a boundary triplet for S^+ and denote the corresponding Weyl function by $\underline{M}$. Then, the triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ defined by*

$$\Gamma_0, \Gamma_1 : S^+ \to \mathbb{C}, \quad \Gamma_0 := \underline{\Gamma}_0, \quad \text{and} \quad \Gamma_1 := \underline{\Gamma}_1 + d\underline{\Gamma}_0,$$

is a boundary triplet for S^+ for each $d \in \mathbb{R}$, and the corresponding Weyl function M fulfils $M = \underline{M} + d$ on $\rho(\ker\Gamma_0)$.

Proof. As $(\underline{\Gamma}_0, \underline{\Gamma}_1)^\top : S^+ \to \mathbb{C}^2$ is surjective also $(\Gamma_0, \Gamma_1)^\top : S^+ \to \mathbb{C}^2$ is surjective. Moreover, we see for $\widehat{f} = \{f, f'\}, \widehat{g} = \{g, g'\} \in S^+$:

$$\begin{aligned}\langle\Gamma_1\widehat{f}, \Gamma_0\widehat{g}\rangle - \langle\Gamma_0\widehat{f}, \Gamma_1\widehat{g}\rangle &= \langle\underline{\Gamma}_1\widehat{f} + d\underline{\Gamma}_0\widehat{f}, \underline{\Gamma}_0\widehat{g}\rangle - \langle\underline{\Gamma}_0\widehat{f}, \underline{\Gamma}_1\widehat{g} + d\underline{\Gamma}_0\widehat{g}\rangle\\ &= \langle\underline{\Gamma}_1\widehat{f}, \underline{\Gamma}_0\widehat{g}\rangle - \langle\underline{\Gamma}_0\widehat{f}, \underline{\Gamma}_1\widehat{g}\rangle = [f', g] - [f, g'],\end{aligned}$$

i.e. the abstract Green identity holds. Hence, $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ is a boundary triplet for S^+ with $\ker\Gamma_0=\ker\underline{\Gamma}_0$. For $\lambda\in\rho(\ker\Gamma_0)$ and $\widehat{f}_\lambda\in\widehat{N}_\lambda(S^+)$ we have by Proposition 2.6(ii)

$$M(\lambda)\Gamma_0\widehat{f}_\lambda=\Gamma_1\widehat{f}_\lambda=\underline{\Gamma}_1\widehat{f}_\lambda+d\underline{\Gamma}_0\widehat{f}_\lambda=\underline{M}(\lambda)\underline{\Gamma}_0\widehat{f}_\lambda+d\underline{\Gamma}_0\widehat{f}_\lambda=(\underline{M}(\lambda)+d)\Gamma_0\widehat{f}_\lambda,$$

i.e. $M(\lambda)=\underline{M}(\lambda)+d$, $\lambda\in\rho(\ker\Gamma_0)$. □

Finally, we show how to combine two Weyl functions, cf. [33, Proposition 4.3]. With this, the realisation of rational functions g symmetric with respect to $\mathbb{R}$ is clear.

Proposition 2.23. *Let $\big(\mathcal{K}_n,[\cdot,\cdot]_n\big)$ and $\big(\mathcal{K}_m,[\cdot,\cdot]_m\big)$ be two Krein spaces with fundamental symmetries J_n and J_m. Then $(\mathcal{K},[\cdot,\cdot])$ is a Krein space, where $\mathcal{K}:=\mathcal{K}_n\oplus\mathcal{K}_m$, $[\cdot,\cdot]:=\langle J\cdot,\cdot\rangle$, and $J:=\left(\begin{smallmatrix}J_n&0\\0&J_m\end{smallmatrix}\right)$. Let two closed symmetric relations $S_n\subseteq\mathcal{K}_n^2$ and $S_m\subseteq\mathcal{K}_m^2$ be given together with their adjoints S_n^+ and S_m^+, respectively. Let $\{\mathbb{C},\Gamma_0^n,\Gamma_1^n\}$ and $\{\mathbb{C},\Gamma_0^m,\Gamma_1^m\}$ be boundary triplets for S_n^+ and S_m^+, such that g_n and g_m are the corresponding Weyl functions, respectively. With the notation $\widehat{f}=\{f,f'\}=\big\{(f_n,f_m)^\top,(f_n',f_m')^\top\big\}\in\mathcal{K}^2$, define the closed symmetric relation S in $\mathcal{K}$ via*

$$S:=\big\{\widehat{f}\in\mathcal{K}^2\,|\,\widehat{f}_n:=\{f_n,f_n'\}\in S_n,\widehat{f}_m:=\{f_m,f_m'\}\in S_m,\,\Gamma_0^n\widehat{f}_n=\Gamma_0^m\widehat{f}_m=0=\Gamma_1^n\widehat{f}+\Gamma_1^m\widehat{f}\big\}.$$

Then the adjoint S^+ of S is given by

$$S^+=\big\{\widehat{f}\in\mathcal{K}^2\,|\,\widehat{f}_n:=\{f_n,f_n'\}\in S_n^+,\widehat{f}_m:=\{f_m,f_m'\}\in S_m^+,\,\Gamma_0^n\widehat{f}_n=\Gamma_0^m\widehat{f}_m\big\}.$$

Moreover, the triplet $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ given by

$$\begin{aligned}\Gamma_0&:S^+\to\mathbb{C},\quad\Gamma_0\widehat{f}=\Gamma_0^n\widehat{f}_n=\Gamma_0^m\widehat{f}_m,\\\Gamma_1&:S^+\to\mathbb{C},\quad\Gamma_1\widehat{f}=\Gamma_1^n\widehat{f}_n+\Gamma_1^m\widehat{f}_m,\end{aligned}$$

is a boundary triplet for S^+ and the corresponding Weyl function M fulfils $M=g_n+g_m$ on $\rho(\ker\Gamma_0)$. Additionally, $\ker\Gamma_0$ is isometrically isomorphic to $\ker\Gamma_0^n\oplus\ker\Gamma_0^m$.

Proof. To calculate S^+ let $\widehat{g}=\{g,g'\}=\{(f_n,f_m)^\top,(f_n',f_m')^\top\}\in S$ and $\widehat{f}=\{f,f'\}=\big\{(f_n,f_m)^\top,(f_n',f_m')^\top\big\}\in S^+$. Then

$$\begin{aligned}0=[f',g]-[f,g']&=[f_n',g_n]_n-[f_n,g_n']_n+[f_m',g_m]_m-[f_m,g_m']_m\\&=\langle\Gamma_1^n\widehat{f}_n,\Gamma_0^n\widehat{g}_n\rangle-\langle\Gamma_0^n\widehat{f}_n,\Gamma_1^n\widehat{g}_n\rangle+\langle\Gamma_1^m\widehat{f}_m,\Gamma_0^m\widehat{g}_m\rangle-\langle\Gamma_0^m\widehat{f}_m,\Gamma_1^m\widehat{g}_m\rangle\\&=-\langle\Gamma_0^n\widehat{f}_n,\Gamma_1^n\widehat{g}_n\rangle-\langle\Gamma_0^m\widehat{f}_m,\Gamma_1^m\widehat{g}_m\rangle\\&=\langle\Gamma_0^n\widehat{f}_n,\Gamma_1^m\widehat{g}_m\rangle-\langle\Gamma_0^m\widehat{f}_m,\Gamma_1^m\widehat{g}_m\rangle\\&=\langle\Gamma_0^n\widehat{f}_n-\Gamma_0^m\widehat{f}_m,\Gamma_1^m\widehat{g}_m\rangle,\end{aligned}$$

i.e. $\Gamma_0^n\widehat{f}_n=\Gamma_0^m\widehat{f}_m$. The first line shows in particular $\{f_n,f_n'\}\in S_n^+$, $\{f_m,f_m'\}\in S_m^+$. Moreover, we have $S\subseteq S^+$ and hence S is symmetric. Obviously, S is also closed. To show the abstract Green identity, let $\widehat{f} = \{f, f'\}, \widehat{g} = \{g, g'\} \in S^+$. It follows that

$$\begin{aligned}
[f', g] - [f, g'] &= \langle f', Jg\rangle - \langle Jf, g'\rangle \\
&= \langle f_n', J_n g_n\rangle_n + \langle f_m', J_m g_m\rangle_m - \langle J_n f_n, g_n'\rangle_n - \langle J_m f_m, g_m'\rangle_m \\
&= \langle \Gamma_1^n \widehat{f}_n, \Gamma_0^n \widehat{g}_n\rangle - \langle \Gamma_0 \widehat{f}_n, \Gamma_1^n \widehat{g}_n\rangle + \langle \Gamma_1^m \widehat{f}_m, \Gamma_0^m \widehat{g}_m\rangle - \langle \Gamma_0^m \widehat{f}_m, \Gamma_1 \widehat{g}_m\rangle \\
&= \langle \Gamma_1^n \widehat{f}_n, \Gamma_0^n \widehat{g}_n\rangle - \langle \Gamma_0 \widehat{f}_n, \Gamma_1^n \widehat{g}_n\rangle + \langle \Gamma_1^m \widehat{f}_m, \Gamma_0^n \widehat{g}_n\rangle - \langle \Gamma_0^n \widehat{f}_n, \Gamma_1 \widehat{g}_m\rangle \\
&= \langle \Gamma_1^n \widehat{f}_n + \Gamma_1^m \widehat{f}_m, \Gamma_0^n \widehat{g}_n\rangle - \langle \Gamma_0 \widehat{f}_n, \Gamma_1^n \widehat{g}_n + \Gamma_1 \widehat{g}_m\rangle \\
&= \langle \Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}\rangle - \langle \Gamma_0 \widehat{f}, \Gamma_1 \widehat{g}\rangle.
\end{aligned}$$

The mapping $\Gamma = (\Gamma_0, \Gamma_1)^\top$ is surjective. Hence, $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for S^+. Moreover, if $\widehat{f}_\lambda \in \widehat{N}_\lambda(S^+)$, $\lambda \in \rho(\ker \Gamma_0)$, we see with Proposition 2.6(ii) for the Weyl function M that

$$M(\lambda)\Gamma_0 \widehat{f}_\lambda = \Gamma_1 \widehat{f}_\lambda = \Gamma_1^n \widehat{f}_{n,\lambda} + \Gamma_1^m \widehat{f}_{m,\lambda} = g_n(\lambda)\Gamma_0^n \widehat{f}_{n,\lambda} + g_m(\lambda)\Gamma_0^m \widehat{f}_{m,\lambda} = (g_n(\lambda) + g_m(\lambda))\Gamma_0 \widehat{f}_\lambda,$$

i.e. $M = g_n + g_m$ on $\rho(\ker \Gamma_0)$. Finally, $\widehat{f} \in \Gamma_0 \subseteq S^+$ if and only if $\widehat{f}_n \in \ker \Gamma_0^n$ and $\widehat{f}_m \in \ker \Gamma_0^m$. Hence, $\ker \Gamma_0$ and $\ker \Gamma_0^n \oplus \ker \Gamma_0^m$ are isometrically isomorphic. □

Remark 2.24. If in the situation of Proposition 2.23 $A_0^n := \ker \Gamma_0^n$ and $A_0^m := \ker \Gamma_0^m$ are operators in $\mathcal{K}_n$ and $\mathcal{K}_m$, respectively, then $A_0 := \ker \Gamma_0$ is an operator in $\mathcal{K}$. Indeed, if $f' \in \operatorname{mul} A_0$, then $\{(0,0)^\top, (f_n', f_m')^\top\} \in A_0$. In particular, $f_n' \in \operatorname{mul} A_0^n = \{0\}$ and $f_m' \in \operatorname{mul} A_0^m = \{0\}$ and thus $f' = 0$.

The next theorem summarises the results of this section and shows in particular that for suitable rational functions g there always exists a realisation. This is a well known result and for example a partial case of the model studied in [13, Theorem 4.10]; cf. also [36, 67, 87].

Theorem 2.25. *Let a rational function g be given in the form*

$$g(\lambda) = c\,\frac{\Pi_{j=1}^m (\lambda - \mu_j)}{\Pi_{k=1}^n (\lambda - \lambda_k)}, \quad \lambda \in \mathbb{C} \backslash \{\lambda_1, \dots, \lambda_n\},$$

with $n \geq m$, $c \in \mathbb{R}$, $\mu_j \neq \lambda_k$, $j = 1, \dots, m$, $k = 1, \dots, n$, such that $\Pi_{j=1}^m (\lambda - \mu_j)$ and $\Pi_{k=1}^n (\lambda - \lambda_k)$ are real-valued. Then there exists $p \geq n$, a matrix $A_0 \in \mathbb{C}^p \times \mathbb{C}^p$, a closed symmetric relation S in $\mathbb{C}^p$ and a boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for S^+ such that $A_0 = \ker \Gamma_0$ and the corresponding Weyl function M fulfils $M = g$.

Proof. Since $n \geq m$ the partial fraction decomposition of g consists only of poles at $\lambda_k \in \mathbb{C}$, $k = 1, \dots, n$, and a constant $d \in \mathbb{R}$ (which may be zero). In particular, there is no pole at ∞. For each k we either have $\lambda_k \in \mathbb{R}$ or there exists λ_l, $l \neq k$, such that $\lambda_k = \overline{\lambda_l}$. Let p denote the sum of the multiplicities of the poles λ_k, $k = 1, \dots, n$, in the partial fraction decomposition of g. Then $p \geq n$ and with the results of this section we find a symmetric relation S in $\mathbb{C}^p$ and a boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for S^+ such that the corresponding Weyl function M fulfils $M = g$. As g has no pole at ∞ we see that $A_0 := \ker \Gamma_0$ is a matrix, cf. Remark 2.24. □

3. Finite Rank Perturbations of Root Subspaces

In order to analyse root subspaces of non-negative operators in Krein spaces under rank one perturbations we first study the influence of perturbations on the number of linearly independent Jordan chains. As this result holds under very general assumptions we present it in Section 3.1 for finite rank perturbations of closed operators. Afterwards, we improve this result for the case of selfadjoint operators under rank one perturbations in Section 3.2.

3.1 Root Subspaces under Finite Rank Perturbations

We present in this section results on finite rank perturbations of closed operators in Krein spaces, which were proven in a slightly more general setting in [18]. The main result of this section is the following theorem.

Theorem 3.1. *Let A and B be closed operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$. Let $\lambda_0 \in \rho(A) \cap \rho(B)$ such that*

$$\dim\big(\operatorname{ran}\big((A-\lambda_0)^{-1} - (B-\lambda_0)^{-1}\big)\big) = k < \infty. \tag{3.1}$$

Then, the following holds for every $\lambda \in \mathbb{C}$.

(i) *If $\ker(A-\lambda)^n$ is finite dimensional for some $n \in \mathbb{N}$, then the same holds for $\ker(B-\lambda)^n$ and*

$$|\dim\ker(A-\lambda)^n - \dim\ker(B-\lambda)^n| \leq k\,n. \tag{3.2}$$

(ii) *If $\ker(A-\lambda)^{n+1}/\ker(A-\lambda)^n$ is finite dimensional for some $n \in \mathbb{N}$, then the same holds for $\ker(B-\lambda)^{n+1}/\ker(B-\lambda)^n$ and*

$$\left|\dim\left(\frac{\ker(A-\lambda)^{n+1}}{\ker(A-\lambda)^n}\right) - \dim\left(\frac{\ker(B-\lambda)^{n+1}}{\ker(B-\lambda)^n}\right)\right| \leq k. \tag{3.3}$$

The proof of Theorem 3.1 is given at the end of this section since we need some auxiliary statements. First, we show in the following example that the estimates in Theorem 3.1 are sharp in a certain sense.

Example 3.2. In $\mathcal{K} = \mathbb{C}^m$ consider a fixed basis $\{e_1, \dots, e_m\}$ and, with respect to this basis, let the linear operators A_1 and B_1 be given via their $m \times m$ matrix-representation

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then A_1 and B_1 satisfy (3.1) with $k = 1$ in the Krein space induced by $J = \operatorname{sip}(m)$ (cf. (2.16)), and we have for $j \leq m$

$$\ker A_1^j = \operatorname{span}\{e_1, \dots, e_j\} \quad \text{and} \quad \ker B_1^j = \{0\}.$$

Hence, the assertions in Theorem 3.1 are sharp for the case $\lambda = 0$ and $k = 1$. In order to obtain sharpness for general $k \in \mathbb{N}$ consider the $(mk \times mk)$-matrices in $\mathbb{C}^{mk}$,

$$A = A_1 \oplus \cdots \oplus A_1 \quad \text{and} \quad B = B_1 \oplus \cdots \oplus B_1.$$

In the following corollary the bounds in Theorem 3.1 are considered in the context of the dimensions of the root subspaces.

Corollary 3.3. *Let A and B be as in Theorem 3.1. Assume that the root subspace $\mathcal{L}_\lambda(A)$ of A at $\lambda \in \mathbb{C}$ is finite dimensional. Then, the following holds.*

(i) *If the maximal length of Jordan chains of A at λ is bounded by p then*

$$|\dim \mathcal{L}_\lambda(A) - \dim \ker(B - \lambda)^p| \leq k\, p.$$

(ii) *If the maximal lengths of Jordan chains of A at λ and Jordan chains of B at λ are bounded by p and q, respectively, then $\mathcal{L}_\lambda(B)$ is finite dimensional and*

$$\left|\dim \mathcal{L}_\lambda(A) - \dim \mathcal{L}_\lambda(B)\right| \leq k\, \max\{p, q\}.$$

Proof. In (i) we have $\mathcal{L}_\lambda(A) = \ker(A-\lambda)^p$. In (ii) we have in addition $\mathcal{L}_\lambda(B) = \ker(B-\lambda)^q$. Then (i) and (ii) follow from (3.2). □

The bound in Corollary 3.3(ii) can be improved if the number k in (3.1) is small compared to the number of linearly independent Jordan chains of A. The following corollary was obtained in [45, 91] for matrices and in [57, Theorem 3] for compact operators. The proof of Corollary 3.4 follows the same arguments as the proof of [91, Corollary 1]. Nevertheless, for the sake of completeness we give the proof here.

Corollary 3.4. *Let A and B be as in Theorem 3.1. Assume that the root subspace $\mathcal{L}_\lambda(A)$ of A at $\lambda \in \mathbb{C}$ is finite dimensional. Choose a basis of $\mathfrak{L}_\lambda(A)$ consisting of linearly independent Jordan chains of A and let $n_1 \geq n_2 \geq \cdots \geq n_l$ be the lengths of these Jordan chains. Then* *we have for $k \leq l$*

$$\dim \mathcal{L}_\lambda(A) - n_1 - n_2 - \cdots - n_k \leq \dim \mathcal{L}_\lambda(B). \tag{3.4}$$

Proof. As $d_j := \dim(\ker(A-\lambda)^j / \ker(A-\lambda)^{j-1}) < \infty$, $j \in \mathbb{N}$, we have $\dim(\ker(B-\lambda)^j / \ker(B-\lambda)^{j-1}) < \infty$ by Theorem 3.1(ii) for every j. Moreover, $d_1 \geq d_2 \geq \ldots \geq 0$ and by assumption there exists $p \in \mathbb{N}$ such that $d_j > 0$ for $j \leq p$ and $d_j = 0$ for $j > p$. Then, $d_1 = l$ and $p = n_1$. In addition, $\dim \mathfrak{L}_\lambda(A) = \sum_{i=1}^{l} n_i = \sum_{i=1}^{p} d_i$. More precisely, for every $m \leq d_1 = l$, we have

$$\sum_{i=1}^{m} n_i = \sum_{i=1}^{l} \min\{m, d_i\}.$$

With $k \leq l$ and Theorem 3.1(ii) it follows

$$\begin{aligned}
\dim \mathfrak{L}_\lambda(B) &= \sum_{j=1}^{\infty} \dim\left(\ker(B-\lambda)^j / \ker(B-\lambda)^{j-1}\right) \\
&\geq \sum_{j=1}^{\infty} \min\left\{\dim\left(\ker(A-\lambda)^j / \ker(A-\lambda)^{j-1}\right) - k, 0\right\} \\
&= \sum_{j=1}^{\infty} \Big(\dim\left(\ker(A-\lambda)^j / \ker(A-\lambda)^{j-1}\right) - \min\{k, d_j\}\Big) \\
&= \dim \mathfrak{L}_\lambda(A) - \sum_{j=1}^{p} \min\{k, d_j\} \\
&= \dim \mathfrak{L}_\lambda(A) - \sum_{j=1}^{k} n_j,
\end{aligned}$$

which shows (3.4). $\square$

Now, we turn to the proof of Theorem 3.1. As a main tool in this proof (and in Section 3.2 and Chapter 4) we construct a space M on which A and B coincide. For this, consider the space $\mathcal{H} := \operatorname{ran}\big((A-\lambda_0)^{-1} - (B-\lambda_0)^{-1}\big)$ which by assumption is k dimensional. Then, $(A-\overline{\lambda}_0)^{-1}$ and $(B-\overline{\lambda}_0)^{-1}$ coincide on $\mathcal{H}^{[\perp]}$. Define

$$M := (A-\overline{\lambda}_0)^{-1}\mathcal{H}^{[\perp]} = (B-\overline{\lambda}_0)^{-1}\mathcal{H}^{[\perp]}. \tag{3.5}$$

With the decomposition $\mathcal{K} = \mathcal{H}^{[\perp]} \dotplus J\mathcal{H}$ for a fundamental symmetry J in $\mathcal{K}$, we see

$$\operatorname{dom} A = (A-\overline{\lambda}_0)^{-1}\mathcal{K} = (A-\overline{\lambda}_0)^{-1}\big(\mathcal{H}^{[\perp]} \dotplus J\mathcal{H}\big) = M \dotplus (A-\overline{\lambda}_0)^{-1}J\mathcal{H},$$

and an analogous calculation holds for $(B-\overline{\lambda}_0)^{-1}$. As $(A-\overline{\lambda}_0)^{-1}$, $(B-\overline{\lambda}_0)^{-1}$, and J are isomorphisms on $\mathcal{K}$, it follows that $M \subseteq \operatorname{dom} A \cap \operatorname{dom} B$ has codimension k in $\operatorname{dom} A$ and

$\operatorname{dom} B$, i.e.

$$\dim(\operatorname{dom} A/M) = \dim(\operatorname{dom} B/M) = k \tag{3.6}$$

and A and B coincide on M.

We show Theorem 3.1 first for the special case $k = 1$. Note that it suffices to prove the result for $\lambda = 0$, otherwise replace A and B by $A - \lambda$ and $B - \lambda$. Theorem 3.1 in this situation is formulated below in Proposition 3.7. As a preparation we state two simple lemmas. The first is an immediate consequence of the fact that A and B coincide on the subspace M.

Lemma 3.5. *Let A and B be as in Theorem 3.1. If $\{x_0, \ldots, x_n\}$ is a Jordan chain of A at λ such that $x_j \in M$ for every $j = 0, \ldots, n$, then $\{x_0, \ldots, x_n\}$ is also a Jordan chain of B at λ.*

The next lemma follows from the fact that for a linear operator B in $\mathcal{K}$ the mapping $x + \ker B \mapsto Bx$, is an isomorphism between $\mathcal{K}/\ker B$ and $\operatorname{ran} B$.

Lemma 3.6. *For a linear operator B in $\mathcal{K}$ the set $\{x_1 + \ker B, \ldots, x_m + \ker B\}$ is linearly independent in $\mathcal{K}/\ker B$ if and only if the set $\{Bx_1, \ldots, Bx_m\}$ is linearly independent in $\mathcal{K}$.*

The next proposition is Theorem 3.1 in the special case $k = 1$ and $\lambda = 0$.

Proposition 3.7. *Let A and B be as in Theorem 3.1 with $k = 1$. Then the following holds.*

(i) *If $\ker A^n$ is finite dimensional for some $n \in \mathbb{N}$, $n \geq 1$, then the same holds for $\ker B^n$ and*

$$|\dim\ker A^n - \dim\ker B^n| \leq n. \tag{3.7}$$

(ii) *If $\ker A^{n+1}/\ker A^n$ is finite dimensional for some $n \in \mathbb{N}$, $n \geq 1$, then the same holds for $\ker B^{n+1}/\ker B^n$ and*

$$\left|\dim\left(\ker A^{n+1}/\ker A^n\right) - \dim\left(\ker B^{n+1}/\ker B^n\right)\right| \leq 1. \tag{3.8}$$

Proof. First, we show (i) for the case $n = 1$, i.e.

$$|\dim\ker A - \dim\ker B| \leq 1. \tag{3.9}$$

Assume that $\ker A$ is finite dimensional and $\dim\ker B > \dim\ker A + 1$. Then there exist $m := \dim\ker A + 2$ linearly independent vectors $\{x_1, \ldots, x_m\}$ in $\ker B$. If $x_j \in M$ then $Ax_j = Bx_j$. So, if $x_j \in M$ for all $j = 1, \ldots, m$ then $\{x_1, \ldots, x_m\} \subseteq \ker A$, a contradiction.

Hence, there exists $1 \leq k_0 \leq m$ such that $x_{k_0} \in \ker B \backslash M$. After reordering we can assume that $k_0 = m$. As $\dim(\operatorname{dom} B/M) = 1$ it is easy to see that there exist $\alpha_k \in \mathbb{C}$ such that

$$z_k := x_k - \alpha_k x_m \in M, \qquad k = 1, \ldots, m-1.$$

Thus $Az_k = Bz_k = 0$ for $k = 1, \ldots, m-1$, and we conclude that $\{z_1, \ldots, z_{m-1}\}$ is a linearly independent set in $\ker A$; a contradiction. Therefore, $\dim\ker B \leq \dim\ker A + 1$

and, in particular, $\ker B$ is finite dimensional. By interchanging A and B we also obtain $\dim\ker A - 1 \leq \dim\ker B$ and hence (3.9) follows.

In the following we prove (ii). Let $n \in \mathbb{N}$, $n \geq 1$, such that $\ker A^{n+1}/\ker A^n$ is finite dimensional and set

$$m := \dim(\ker A^{n+1}/\ker A^n) + 2. \tag{3.10}$$

Assume that the set $\{x_{1,n}+\ker B^n, \ldots, x_{m,n}+\ker B^n\}$ is linearly independent in the quotient space $\ker B^{n+1}/\ker B^n$. For $k = 1, \ldots, m$ construct the following Jordan chains of B at 0:

$$x_{k,0} := B^n x_{k,n}, \quad x_{k,1} := B^{n-1}x_{k,n}, \quad \ldots, \quad x_{k,n-1} := Bx_{k,n}.$$

Then, $x_{k,0} \in \ker B$ for $k = 1, \ldots, m$ and, applying Lemma 3.6 to B^n it follows that

$$\{x_{1,0}, \ldots, x_{m,0}\} \text{ is a linearly independent set in } \ker B. \tag{3.11}$$

Define the index set $\mathfrak{I}$ by

$$\mathfrak{I} := \big\{(k,j) \,|\; x_{k,j} \notin M, 1 \leq k \leq m, 0 \leq j \leq n\big\}.$$

The set $\mathfrak{I}$ is not empty. Otherwise $\{x_{k,0}, \ldots x_{k,n}\} \subseteq M$ for every $1 \leq k \leq m$ and, by Lemma 3.5, these m (linearly independent) Jordan chains of B at 0 of length $n+1$ are as well (linearly independent) Jordan chains of A at 0 of length $n+1$, a contradiction to (3.10). Hence, we set

$$h := \min\big\{j \,|\; (k,j) \in \mathfrak{I} \text{ for some } k \text{ with } 1 \leq k \leq m\big\}.$$

After a reordering of the indices, we assume that $(m,h) \in \mathfrak{I}$, i.e. $x_{m,h} \notin M$. Then,

$$j < h \text{ implies } x_{k,j} \in M \text{ for all } k = 1, \ldots, m. \tag{3.12}$$

In what follows we construct $m-1$ elements $z_1, \ldots, z_{m-1}$ in $\ker A^{n+1}$ such that

$$\{z_1 + \ker A^n, \ldots, z_{m-1} + \ker A^n\}$$

is linearly independent in $\ker A^{n+1}/\ker A^n$, which is a contradiction to (3.10). We consider three different cases.

Case I: $h = n$**.** Since $x_{m,n} \notin M$, there exist $\alpha_{k,n} \in \mathbb{C}$ such that

$$z_k := x_{k,n} - \alpha_{k,n}x_{m,n} \in M \cap \ker B^{n+1} \quad \text{for } k = 1, \ldots, m-1.$$

From (3.12) it follows that, for every $k = 1, \ldots, m-1$, the Jordan chain

$$\{x_{k,0} - \alpha_{k,n}x_{m,0}, \ldots, x_{k,n-1} - \alpha_{k,n}x_{m,n-1}, z_k\}$$

of B at 0 is contained in M. Then, by Lemma 3.5 these also are $m-1$ (linearly independent) Jordan chains of A at 0 of length n. In particular, the set $\{z_1 + \ker A^n, \ldots, z_{m-1} + \ker A^n\}$ is linearly independent in $\ker A^{n+1}/\ker A^n$.

Case II: $h = n-1$**.** Since $x_{m,n-1} \notin M$, there exist $\alpha_{k,n-1} \in \mathbb{C}$ such that

$$v_{k,n-1} := x_{k,n-1} - \alpha_{k,n-1} x_{m,n-1} \in M \cap \ker B^n \quad \text{for } k = 1, \dots, m-1.$$

Let $w_{k,n} := x_{k,n} - \alpha_{k,n-1} x_{m,n} \in \ker B^{n+1}$ for $k = 1, \dots, m-1$, and choose $\alpha_{k,n} \in \mathbb{C}$ such that

$$z_k := w_{k,n} - \alpha_{k,n} x_{m,n-1} \in M \cap \ker B^{n+1} \quad \text{for } \; k = 1, \dots, m-1.$$

Since $z_k \in M$ and $v_{k,n-1} \in M$, $k = 1, \dots, m-1$, we conclude from $Bw_{k,n} = v_{k,n-1}$ together with (3.12) that

$$\begin{aligned}
A^{n+1} z_k &= A^n A z_k = A^n B z_k \\
&= A^n B(w_{k,n} - \alpha_{k,n} x_{m,n-1}) = A^n (v_{k,n-1} - \alpha_{k,n} x_{m,n-2}) \\
&= A^{n-1} B(v_{k,n-1} - \alpha_{k,n} x_{m,n-2}) \\
&= A^{n-1} B(x_{k,n-1} - \alpha_{k,n-1} x_{m,n-1} - \alpha_{k,n} x_{m,n-2}) \\
&= A^{n-1} (x_{k,n-2} - \alpha_{k,n-1} x_{m,n-2} - \alpha_{k,n} x_{m,n-3}) \\
&\;\;\vdots \\
&= A^2 (x_{k,1} - \alpha_{k,n-1} x_{m,1} - \alpha_{k,n} x_{m,0}) \\
&= AB(x_{k,1} - \alpha_{k,n-1} x_{m,1} - \alpha_{k,n} x_{m,0}) \\
&= A(x_{k,0} - \alpha_{k,n-1} x_{m,0}) = B(x_{k,0} - \alpha_{k,n-1} x_{m,0}) = 0,
\end{aligned}$$

and $A^n z_k = x_{k,0} - \alpha_{k,n-1} x_{m,0} \neq 0$ for all $k = 1, \dots, m-1$. By (3.11) the set

$$\{x_{1,0} - \alpha_{1,n-1} x_{m,0}, \dots, x_{m-1,0} - \alpha_{m-1,n-1} x_{m,0}\}$$

is linearly independent. Then by Lemma 3.6 applied to A^n it follows that the set $\{z_1 + \ker A^n, \dots, z_{m-1} + \ker A^n\}$ is linearly independent in $\ker A^{n+1} / \ker A^n$.

Case III: $0 \leq h \leq n-2$**.** In this case we construct as in Case II two sets of vectors

$$\big\{v_{k,j} \in M \cap \ker B^{j+1} : k = 1, \dots, m-1, j = h, \dots, n-1\big\}, \tag{3.13}$$

and

$$\big\{w_{k,j+1} \in \ker B^{j+2} : k = 1, \dots, m-1, j = h, \dots, n-1\big\}. \tag{3.14}$$

By assumption, $x_{m,h} \notin M$. We start the construction with $j = h$, that is, with the definition of the vectors $v_{k,h}$ and $w_{k,h+1}$ for $k = 1, \dots, m-1$: There exist $\alpha_{k,h} \in \mathbb{C}$ such that

$$v_{k,h} := x_{k,h} - \alpha_{k,h} x_{m,h} \in M \cap \ker B^{h+1} \quad \text{for } k = 1, \dots, m-1.$$

Using the same coefficients $\alpha_{k,h} \in \mathbb{C}$, let

$$w_{k,h+1} := x_{k,h+1} - \alpha_{k,h} x_{m,h+1} \in \ker B^{h+2} \quad \text{for } \; k = 1, \dots, m-1.$$

Note that $Bw_{k,h+1} = v_{k,h}$ for $k = 1, \dots, m-1$. The vectors $v_{k,j}$ and $w_{k,j+1}$, $k = 1, \dots, m-1$, are defined inductively for $j = h+1, \dots, n-1$, in the following way: Fix $j \in \{h+1, \dots, n-1\}$ and assume that we have constructed $v_{k,j-1} \in M \cap \ker B^j$ and $w_{k,j} \in \ker B^{j+1}$ for $k = 1, \dots, m-1$ with $Bw_{k,j} = v_{k,j-1}$. Then there exist $\alpha_{k,j} \in \mathbb{C}$ such that

$$v_{k,j} := w_{k,j} - \alpha_{k,j} x_{m,h} \in M \cap \ker B^{j+1} \quad \text{for} \quad k = 1, \dots, m-1.$$

Also, define

$$w_{k,j+1} := x_{k,j+1} - \sum_{i=0}^{j-h} \alpha_{k,h+i} x_{m,j-i+1} \in \ker B^{j+2} \quad \text{for} \quad k = 1, \dots, m-1.$$

A straightforward computation shows $Bw_{k,j+1} = v_{k,j}$ for $k = 1, \dots, m-1$. So, we have constructed the sets in (3.13) and (3.14).

Finally, observe that there also exist $\alpha_{k,n} \in \mathbb{C}$ such that

$$z_k := w_{k,n} - \alpha_{k,n} x_{m,h} \in M \cap \ker B^{n+1} \quad \text{for} \quad k = 1, \dots, m-1.$$

Hence,

$$\begin{aligned}
Az_k &= Bz_k = B(w_{k,n} - \alpha_{k,n} x_{m,h}) = v_{k,n-1} - \alpha_{k,n} x_{m,h-1}, \\
A^2 z_k &= A(v_{k,n-1} - \alpha_{k,n} x_{m,h-1}) \\
&= B(v_{k,n-1} - \alpha_{k,n} x_{m,h-1}) \\
&= B(w_{k,n-1} - \alpha_{k,n-1} x_{m,h} - \alpha_{k,n} x_{m,h-1}) \\
&= v_{k,n-2} - \alpha_{k,n-1} x_{m,h-1} - \alpha_{k,n} x_{m,h-2},
\end{aligned}$$

and, in the same way, we show that

$$A^{n-h} z_k = \; v_{k,h} - \sum_{i=1}^{n-h} \alpha_{k,h+i} x_{m,h-i},$$

where $x_{m,l} := 0$ if $l < 0$. Also, observe that

$$\begin{aligned}
A^{n-h+1} z_k &= A(v_{k,h} - \sum_{i=1}^{n-h} \alpha_{k,h+i} x_{m,h-i}) \\
&= B(v_{k,h} - \sum_{i=1}^{n-h} \alpha_{k,h+i} x_{m,h-i}) \\
&= B(x_{k,h} - \alpha_{k,h} x_{m,h} - \sum_{i=1}^{n-h} \alpha_{k,h+i} x_{m,h-i}) \\
&= x_{k,h-1} - \sum_{i=0}^{n-h} \alpha_{k,h+i} x_{m,h-i-1}, \\
A^{n-h+2} z_k &= A(x_{k,h-1} - \sum_{i=0}^{n-h} \alpha_{k,h+i} x_{m,h-i-1})
\end{aligned}$$

$$
\begin{aligned}
&= B(x_{k,h-1} - \sum_{i=0}^{n-h} \alpha_{k,h+i} x_{m,h-i-1}) \\
&= x_{k,h-2} - \sum_{i=0}^{n-h} \alpha_{k,h+i} x_{m,h-i-2}, \\
&\ \vdots \\
A^n z_k &= x_{k,0} - \alpha_{k,h} x_{m,0}, \qquad \text{and} \\
A^{n+1} z_k &= 0.
\end{aligned}
$$

Furthermore, as in (3.11) the set $\{x_{1,0} - \alpha_{1,h} x_{m,0}, \ldots, x_{m-1,0} - \alpha_{m-1,h} x_{m,0}\}$ is linearly independent in $\ker A$. Applying Lemma 3.6 to A^n it follows that the set $\{z_1 + \ker A^n, \ldots, z_{m-1} + \ker A^n\}$ is linearly independent in $\ker A^{n+1}/\ker A^n$.

Summing up, we have shown in Cases I-III above that there exists a linearly independent set $\{z_1 + \ker A^n, \ldots, z_{m-1} + \ker A^n\}$ in $\ker A^{n+1}/\ker A^n$, which contradicts (3.10). Therefore,

$$
\dim(\ker B^{n+1}/\ker B^n) \leq \dim(\ker A^{n+1}/\ker A^n) + 1,
$$

and, in particular, $\ker B^{n+1}/\ker B^n$ is finite dimensional. Then, (3.8) follows by interchanging A and B. Finally, (3.7) is a consequence of (3.9) and repeated applications of (3.8). $\square$

Before proving Theorem 3.1 we will improve the upper bound in Proposition 3.7(ii) for a particular class of rank one perturbations.

Assume that A is a closed operator in $\mathcal{K}$ and M is a linear subspace in $\operatorname{dom} A$ such that $\dim\big(\operatorname{dom} A/M\big) = k$. Then, there exist linearly independent vectors $x_1, \ldots, x_k \in (\operatorname{dom} A)\backslash M$ such that

$$
\operatorname{dom} A = M \dot{+} \operatorname{span}\{x_1, \ldots, x_k\}.
$$

We define the restrictions

$$
A_p := A \upharpoonright \big(M \dot{+} \operatorname{span}\{x_1, \ldots, x_p\}\big), \qquad 1 \leq p \leq k.
$$

Lemma 3.8. *Given $2 \leq p \leq k$, if $\ker A_p^{n+1}/\ker A_p^n$ is finite dimensional for some $n \in \mathbb{N}$, then the same holds for $\ker A_{p-1}^{n+1}/\ker A_{p-1}^n$ and*

$$
\dim\left(\frac{\ker A_p^{n+1}}{\ker A_p^n}\right) - 1 \leq \dim\left(\frac{\ker A_{p-1}^{n+1}}{\ker A_{p-1}^n}\right) \leq \dim\left(\frac{\ker A_p^{n+1}}{\ker A_p^n}\right).
$$

Proof. By Proposition 3.7 only the second inequality needs to be proved. Assume that $\dim\big(\ker A_p^{n+1}/\ker A_p^n\big) = i < \infty$ and that the set $\{z_1 + \ker A_{p-1}^n, \ldots, z_{i+1} + \ker A_{p-1}^n\}$ is linearly independent in $\ker A_{p-1}^{n+1}/\ker A_{p-1}^n$. Then, since $\ker A_{p-1}^{n+1} \subseteq \ker A_p^{n+1}$, there exist

$\alpha_1, \ldots, \alpha_{i+1} \in \mathbb{C}$ (not all equal to zero) such that

$$z := \alpha_1 z_1 + \ldots + \alpha_{i+1} z_{i+1} \in \ker A_p^n.$$

Together with $z \in \operatorname{dom} A_{p-1}^{n+1} \subseteq \operatorname{dom} A_{p-1}^n$ we conclude $z \in \ker A_{p-1}^n$, a contradiction, and Lemma 3.8 is shown. □

Proof of Theorem 3.1. By assumption, A and B satisfy (3.1) and we have by (3.6)

$$\dim\bigl(\operatorname{dom} A/M\bigr) = \dim\bigl(\operatorname{dom} B/M\bigr) = k.$$

Then there exist linearly independent vectors $x_1, \ldots, x_k \in (\operatorname{dom} A)\backslash M$ and $y_1, \ldots, y_k \in (\operatorname{dom} B)\backslash M$ such that

$$\operatorname{dom} A = M \dot{+} \operatorname{span}\{x_1, \ldots, x_k\} \quad \text{and} \quad \operatorname{dom} B = M \dot{+} \operatorname{span}\{y_1, \ldots, y_k\}.$$

Also, we can assume that $\operatorname{span}\{x_1, \ldots, x_k\} \cap \operatorname{span}\{y_1, \ldots, y_k\} = \{0\}$ (otherwise M can be enlarged). Next, consider the restrictions

$$A_p := A \upharpoonright \left(M \dot{+} \operatorname{span}\{x_1, \ldots, x_p\}\right), \qquad 1 \leq p \leq k,$$

and

$$B_q := B \upharpoonright \left(M \dot{+} \operatorname{span}\{y_1, \ldots, y_q\}\right), \qquad 1 \leq q \leq k.$$

Clearly $A = A_k$ and $B = B_k$. As mentioned before, it is sufficient to prove Theorem 3.1 for $\lambda = 0$. Let $\ker A^{n+1}/\ker A^n$ be finite dimensional for some $n \in \mathbb{N}$, $n \geq 1$. Applying repeatedly Lemma 3.8 to $A = A_k, A_{k-1}, \ldots, A_2$, we see that $\ker A_1^{n+1}/\ker A_1^n$ is finite dimensional and

$$\dim\left(\frac{\ker A^{n+1}}{\ker A^n}\right) - (k-1) \leq \dim\left(\frac{\ker A_1^{n+1}}{\ker A_1^n}\right) \leq \dim\left(\frac{\ker A^{n+1}}{\ker A^n}\right). \tag{3.15}$$

The operators A_1 and B_1 satisfy (3.1) with $k = 1$. Hence, by Proposition 3.7, $\ker B_1^{n+1}/\ker B_1^n$ is finite dimensional and

$$\left|\dim\left(\ker A_1^{n+1}/\ker A_1^n\right) - \dim\left(\ker B_1^{n+1}/\ker B_1^n\right)\right| \leq 1. \tag{3.16}$$

Similarly, repeated application of Lemma 3.8 to the operators $B_2, B_3, \ldots, B_k = B$ shows that $\ker B^{n+1}/\ker B^n$ is finite dimensional and

$$\dim\left(\frac{\ker B^{n+1}}{\ker B^n}\right) - (k-1) \leq \dim\left(\frac{\ker B_1^{n+1}}{\ker B_1^n}\right) \leq \dim\left(\frac{\ker B^{n+1}}{\ker B^n}\right). \tag{3.17}$$

Then, with (3.15), (3.16), and (3.17)

$$\begin{aligned}\dim\left(\ker A^{n+1}/\ker A^n\right) &- \dim\left(\ker B^{n+1}/\ker B^n\right)\\ &\geq \dim\left(\ker A_1^{n+1}/\ker A_1^n\right) - \dim\left(\ker B^{n+1}/\ker B^n\right)\\ &\geq \dim\left(\ker B_1^{n+1}/\ker B_1^n\right) - 1 - \dim\left(\ker B^{n+1}/\ker B^n\right)\\ &\geq -(k-1) - 1 = -k.\end{aligned}$$

An analogous calculation for the upper bound shows

$$\dim\left(\ker A^{n+1}/\ker A^n\right) - \dim\left(\ker B^{n+1}/\ker B^n\right) \leq k,$$

which yields

$$\left|\dim\left(\ker A^{n+1}/\ker A^n\right) - \dim\left(\ker B^{n+1}/\ker B^n\right)\right| \leq k,$$

and assertion (ii) in Theorem 3.1 holds. Finally, Theorem 3.1(i) follows from

$$|\dim\ker A - \dim\ker B| \leq k,$$

which is shown in a similar way as in the proof of Proposition 3.7 and a repeated application of (3.3). □

3.2 Root Subspaces at Real Points under Rank One Perturbations

We saw in Section 3.1 how root subspaces of closed operators behave under finite rank perturbations. We now focus on rank one perturbations of selfadjoint operators and root subspaces at real points of the spectrum. This allows us to investigate the structure of the root subspaces more precisely. Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that (2.5) holds for some real number $\lambda_0 \in \rho(A)\cap\rho(B)\cap\mathbb{R}$. By Proposition 2.10 we can represent the resolvent difference by

$$C := (B-\lambda_0)^{-1} - (A-\lambda_0)^{-1} = \frac{1}{M_A(\lambda_0)}[\cdot,\varphi_A]\varphi_A. \tag{3.18}$$

Then C is selfadjoint with respect to $[\cdot,\cdot]$ with one-dimensional range. Set $\mathcal{H} := \operatorname{span}\{\varphi_A\}$. As before $(B-\lambda_0)^{-1}$ and $(A-\lambda_0)^{-1}$ coincide on $\{\varphi_A\}^{[\perp]}$ and the space M in (3.5) takes the form

$$M = (A-\lambda_0)^{-1}\{\varphi_A\}^{[\perp]} = (B-\lambda_0)^{-1}\{\varphi_A\}^{[\perp]} \tag{3.19}$$

with $M \subseteq \operatorname{dom} A \cap \operatorname{dom} B$. For $y \in M$ there exists $x \in \{\varphi_A\}^{[\perp]}$ such that $y = (A-\lambda_0)^{-1}x = (B-\lambda_0)^{-1}x$ and we conclude

$$Ay = x + \lambda_0(A-\lambda_0)^{-1}x = x + \lambda_0(B-\lambda_0)^{-1}x = By.$$

Thus, A and B coincide on M and their domains decompose as

$$\begin{aligned}\operatorname{dom} A &= (A-\lambda_0)^{-1}\mathcal{K} = (A-\lambda_0)^{-1}\big(\{\varphi_A\}^{[\perp]} \oplus \operatorname{span}\{J\varphi_A\}\big) = M \dot{+} \operatorname{span}\{f_A\},\\ \operatorname{dom} B &= (B-\lambda_0)^{-1}\mathcal{K} = (B-\lambda_0)^{-1}\big(\{\varphi_A\}^{[\perp]} \oplus \operatorname{span}\{J\varphi_A\}\big) = M \dot{+} \operatorname{span}\{f_B\},\end{aligned}$$

where J is some fundamental symmetry in the Krein space $\mathcal{K}$ and $f_A := (A-\lambda_0)^{-1}J\varphi_A \neq 0$ and $f_B := (B-\lambda_0)^{-1}J\varphi_A \neq 0$. Hence for $x, y \in \operatorname{dom} A$ (or $x, y \in \operatorname{dom} B$) with $y \notin M$ there exists $\alpha \in \mathbb{C}$ such that

$$x - \alpha y \in M.$$

Moreover, for $y \in \operatorname{dom} A \cap \operatorname{dom} B$, we have

$$Ay = By \Leftrightarrow y \in M \Leftrightarrow (A-\lambda_0)y = (B-\lambda_0)y \in \{\varphi_A\}^{[\perp]} \Leftrightarrow (A-\lambda_0)y = (B-\lambda_0)y \in \ker C. \tag{3.20}$$

Lemma 3.9. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that (2.5) holds for some real number $\lambda_0 \in \rho(A)\cap\rho(B)\cap\mathbb{R}$. For some $\mu \in \mathbb{R}$ assume $\dim \mathfrak{L}_\mu(A) < \infty$ and let $\dim\ker(A-\mu) = \dim\ker(B-\mu)$. Then $\ker(A-\mu) = \ker(B-\mu)$.*

Proof. Set $m := \dim\ker(A-\mu) = \dim\ker(B-\mu)$ and assume $\ker(A-\mu) \neq \ker(B-\mu)$. We choose linearly independent vectors $x_1, \ldots, x_m \in \mathcal{K}$, such that $\ker(A-\mu) = \operatorname{span}\{x_1, \ldots, x_m\}$. Then there exists $j \in \{1, \ldots, m\}$ with $x_j \notin M$. After reordering we can assume $x_m \notin M$. Hence, we find $\alpha_1, \ldots, \alpha_{m-1} \in \mathbb{C}$ such that

$$a_j := x_j - \alpha_j x_m \in M, \quad j = 1, \ldots, m-1,$$

and $\{a_1, \ldots, a_{m-1}\} \subseteq \ker(B-\mu)$. Choose $b_m \in \mathcal{K}$ with $\ker(B-\mu) = \operatorname{span}\{a_1, \ldots, a_{m-1}, b_m\}$. Then $b_m \notin M$ and thus $(B-\lambda_0)b_m \notin \{\varphi_A\}^{[\perp]}$. By (3.18) there exists $\beta \neq 0$ with

$$\begin{aligned}0 \neq \beta\varphi_A = C(B-\lambda_0)b_m &= (B-\lambda_0)^{-1}(B-\lambda_0)b_m - (A-\lambda_0)^{-1}(B-\lambda_0)b_m\\ &= b_m - (\mu-\lambda_0)(A-\lambda_0)^{-1}b_m.\end{aligned}$$

This means $(\mu-\lambda_0)(A-\lambda_0)^{-1}b_m = b_m - \beta\varphi_A \in \operatorname{dom} A$ and

$$(\mu-\lambda_0)b_m = (A-\lambda_0)(b_m - \beta\varphi_A) = (A-\mu)(b_m-\beta\varphi_A) + (\mu-\lambda_0)(b_m-\beta\varphi_A),$$

i.e. $(A-\mu)(b_m-\beta\varphi_A) = (\mu-\lambda_0)\beta\varphi_A$. This yields

$$0 = [b_m - \beta\varphi_A, (A-\mu)x_m] = [(A-\mu)(b_m-\beta\varphi_A), x_m] = (\mu-\lambda_0)\beta[\varphi_A, x_m].$$

Hence,

$$[\varphi_A, (A-\lambda_0)x_m] = [\varphi_A, (\mu-\lambda_0)x_m] = (\mu-\lambda_0)[\varphi_A, x_m] = 0,$$

implying $x_m \in M$ by (3.20), which is a contradiction. □

Proposition 3.10. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such* *that* (2.5) *holds for some real number* $\lambda_0 \in \rho(A) \cap \rho(B) \cap \mathbb{R}$. *For some* $\mu \in \mathbb{R}$ *let* $\dim \mathfrak{L}_\mu(A) < \infty$ *and* $\dim \mathfrak{L}_\mu(B) < \infty$. *Assume that* $\mathcal{K}$ *admits the decomposition*

$$\mathcal{K} = \mathfrak{L}_\mu(A)[\dot{+}]\mathfrak{L}_\mu(A)^{[\perp]} \tag{3.21}$$

into two Krein spaces $(\mathfrak{L}_\mu(A), [\cdot,\cdot])$ *and* $(\mathfrak{L}_\mu(A)^{[\perp]}, [\cdot,\cdot])$.

(i) *Let* $\mathfrak{L}_\mu(A) \subseteq \mathfrak{L}_\mu(B)$, $\ker(A-\mu) = \ker(B-\mu)$ *and* $A \upharpoonright \mathfrak{L}_\mu(A) = B \upharpoonright \mathfrak{L}_\mu(A)$. *Then* $\mathfrak{L}_\mu(A) = \mathfrak{L}_\mu(B)$.

(ii) *Let* A *and* B *have Jordan chains at* μ *of length at most* 2 *and* $\dim\ker(A-\mu) = \dim\ker(B-\mu)$. *Then*

$$\dim \mathfrak{L}_\mu(A) = \dim \mathfrak{L}_\mu(B).$$

Proof. (i) The subspaces $(\mathfrak{L}_\mu(A), [\cdot,\cdot])$ and $(\mathfrak{L}_\mu(A)^{[\perp]}, [\cdot,\cdot])$ are invariant under A and also under B, since A and B coincide on $\mathfrak{L}_\mu(A)$. Hence, we obtain decompositions of A and B with respect to (3.21) into

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \quad B = \begin{pmatrix} A_0 & 0 \\ 0 & B_1 \end{pmatrix}$$

with $\mu \in \rho(A_1)$, A_1, B_1 being selfadjoint operators in $(\mathfrak{L}_\mu(A)^{[\perp]}, [\cdot,\cdot])$. Moreover, the selfadjoint operator $(B_1-\lambda_0)^{-1} - (A_1-\lambda_0)^{-1}$ in $(\mathfrak{L}_\mu(A)^{[\perp]}, [\cdot,\cdot])$ has one-dimensional range. Assume $\mathfrak{L}_\mu(A) \neq \mathfrak{L}_\mu(B)$. By assumption we have $\mathfrak{L}_\mu(A) \subseteq \mathfrak{L}_\mu(B)$ and $\ker(A-\mu) = \ker(B-\mu)$. Hence,

$$p := \min\left\{q \,\middle|\, \ker(A-\mu)^q \subsetneqq \ker(B-\mu)^q\right\}$$

is well-defined and $p \geq 2$. Consequently there exists a Jordan chain $\{b_0, \ldots, b_{p-1}\}$ of B at μ of length p such that $b_i \in \mathfrak{L}_\mu(A)$, $i = 0, \ldots, p-2$, and $b_{p-1} \notin \mathfrak{L}_\mu(A)$. With respect to (3.21), we have the representations $y = \left(y^0, 0\right)^\top$ for every $y \in \operatorname{span}\mathfrak{B}$ and $b_{p-1} = \left(b_{p-1}^0, b_{p-1}^1\right)^\top$, where $b_{p-1}^1 \neq 0$. Then

$$\begin{pmatrix} b_{p-2}^0 \\ 0 \end{pmatrix} = b_{p-2} = (B-\mu)b_{p-1} = \begin{pmatrix} (A_0-\mu)b_{p-1}^0 \\ (B_1-\mu)b_{p-1}^1 \end{pmatrix},$$

which gives $b_{p-1}^1 \in \ker(B-\mu)$. But $b_{p-1}^1 \in \mathfrak{L}_\mu(A)^{[\perp]}$ which contradicts $\ker(A-\mu) = \ker(B-\mu)$.

(ii) By Lemma 3.9 we have $\ker(A-\mu) = \ker(B-\mu)$. Assume $\dim \mathfrak{L}_\mu(B) < \dim \mathfrak{L}_\mu(A)$. Then, $n := \dim\left(\ker(B-\mu)^2/\ker(B-\mu)\right) = \dim\left(\ker(A-\mu)^2/\ker(A-\mu)\right) - 1$ by Theorem 3.1. Let $\{y_1, \ldots, y_n\} \in \ker(A-\mu)^2 \setminus \ker(A-\mu)$ be n linearly independent vectors. We set $x_j := (A-\mu)y_j$, $j = 1, \ldots, n$, and complete these linearly independent vectors to a basis

$$\{x_1, \ldots, x_n, x_{n+1}, \ldots, x_m\}$$

of $\ker(A-\mu)$. The set $\{x_1, \ldots x_m, y_1, \ldots, y_n\}$ is a basis of $\mathfrak{L}_\mu(A)$ with $x_1, \ldots, x_m \in M$ and there exists $k \in \{1, \ldots, n\}$ such that $y_k \notin M$. After reordering we can assume $y_n \notin M$.

Hence, we find $\alpha_j \in \mathbb{C}$ such that $v_j := y_j - \alpha_j y_n \in M$, $j = 1\ldots, n-1$. Set

$$\begin{aligned} u_j &:= (A-\mu)v_j = x_j - \alpha_j x_1 \in M, \quad && j = 1,\ldots,n-1, \quad \text{and} \\ u_j &:= x_j \in M, && j = n+1,\ldots,m. \end{aligned}$$

Then,

$$\begin{aligned} (B-\mu)v_j &= (A-\mu)v_j = u_j, \quad && j = 1,\ldots,n-1, \quad \text{and} \\ (B-\mu)u_j &= (A-\mu)u_j = 0, && j = 1,\ldots,m. \end{aligned}$$

In particular, $\{u_1,\ldots,u_m,v_1,\ldots,v_{n-1}\}$ is a basis of $\mathfrak{L}_\mu(B)$ and hence $\mathfrak{L}_\mu(B) \subseteq \mathfrak{L}_\mu(A)$. The above considerations show $A \upharpoonright \mathfrak{L}_\mu(B) = B \upharpoonright \mathfrak{L}_\mu(B)$ which yields $\mathfrak{L}_\mu(B) = \mathfrak{L}_\mu(A)$ by (ii), a contradiction. Hence, $\dim \mathfrak{L}_\mu(B) \geq \dim \mathfrak{L}_\mu(A)$. The other inequality is shown analogously. □

Remark 3.11. (i) Observe that in Lemma 3.9 and Proposition 3.10 it is essential to assume A and B to be selfadjoint in a Krein space. For example consider

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is easy to check that there exists no Krein space inner product $[\cdot,\cdot]$ in $\mathbb{C}^2$ such that A and B are both selfadjoint matrices with respect to $[\cdot,\cdot]$. We see $\dim \operatorname{ran}(A-B) = 1$ but for $\mu = 1$ the statements in Lemma 3.9 and Proposition 3.10(ii) are not true.

(ii) Proposition 3.10(ii) does not imply $\mathfrak{L}_\mu(A) = \mathfrak{L}_\mu(B)$. Indeed, consider the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{pmatrix}, \quad B - A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where J denotes a fundamental symmetry in $\mathbb{C}^3$. Then, A, B, and $\mu = 0$ fulfil the assumptions of Proposition 3.10 and $\ker A = \ker B = \operatorname{span}\{(1,0,0)^\top\}$, but

$$\mathfrak{L}_0(B) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{L}_0(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(iii) Even if $\mathfrak{L}_\mu(A) = \mathfrak{L}_\mu(B)$ holds true, this does not imply $A \upharpoonright \mathfrak{L}_\mu(A) = B \upharpoonright \mathfrak{L}_\mu(A)$. Indeed, consider the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B - A = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the fundamental symmetry

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, A, B, and $\mu = 0$ fulfil the assumptions of Proposition 3.10 and $\mathfrak{L}_0(A) = \mathfrak{L}_0(B) = \mathbb{C}^4$, but $Ae_2 = e_1$, $Be_2 = e_3$, $Ae_4 = e_3$, $Be_4 = e_1$, where $\{e_1, \dots, e_4\}$ denotes the standard basis of $\mathbb{C}^4$.

(iv) The assumption on the length of Jordan chains in Proposition 3.10(ii) is as well important. To see this consider the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \ B = \frac{1}{5}\begin{pmatrix} 0 & 2 & -6 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -6 & 3 & 6 \\ 0 & -3 & -6 & -12 \end{pmatrix}, \ B - A = \frac{1}{5}\begin{pmatrix} 0 & -3 & -6 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & -6 & -12 & 6 \\ 0 & -3 & -6 & 3 \end{pmatrix},$$

and the fundamental symmetry

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Note that the Jordan form of B is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{9}{5} \end{pmatrix}.$$

Then, A, B, and $\mu = 0$ fulfil the assumptions of Proposition 3.10 and we have $\mathfrak{L}_0(A) = \operatorname{span}\{e_1, e_2\}$ and $\mathfrak{L}_0(B) = \operatorname{span}\{e_1, (0,0,2,-1)^\top, (0,3,0,-2)^\top\}$. In particular, we see $\dim \mathfrak{L}_0(B) > \dim \mathfrak{L}_0(A)$.

4. Root Subspaces of Non-negative Operators

In Chapter 3 we investigated root subspaces under perturbation in a rather general setting. In this chapter we study non-negative operators in Krein spaces under rank one perturbations and present results on the change of the structure of root subspaces. Since 0 is an exceptional point in the spectrum of a non-negative operator the results for root subspaces and Jordan chains at $\mu = 0$ are given in Section 4.1 and the results for $\mu \neq 0$ are given in Section 4.2.

4.1 Root Subspaces at $\mu = 0$

Let A be a non-negative operator in the Krein space $(\mathcal{K}, [\cdot,\cdot])$. Then

$$x \in \operatorname{dom} A,\ [Ax, x] = 0 \quad \text{implies} \quad x \in \ker A. \tag{4.1}$$

Indeed, the application of the Cauchy-Bunyakowski inequality to the semi-definite inner product $[A\cdot,\cdot]$ gives $|[Ax, y]|^2 \leq [Ax, x][Ay, y]$ for all $x, y \in \operatorname{dom} A$, and (4.1) follows. If 0 is an isolated eigenvalue of A with finite algebraic multiplicity we have by Proposition 1.12 the decomposition $\mathcal{K} = \mathfrak{L}_0(A)[\dot{+}]\mathfrak{L}_0(A)^{[\perp]}$.

First, we show well-known bounds on the lengths of the Jordan chains of A and B at 0.

Lemma 4.1. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that (2.5) holds for some $\lambda_0 \in \rho(A) \cap \rho(B)$ and let A be non-negative. Then the following holds.*

(i) *A has Jordan chains at 0 of length at most 2.*

(ii) *B has Jordan chains at 0 of length at most 4.*

Proof. Assertion (i) is a consequence of Remark 1.15. In order to show (ii) assume that B has a Jordan chain $\{x_0, \ldots, x_4\}$ at 0 of length 5 and let M be given as in (3.19). Then

$$[x_2, x_1] = [B^2 x_4, x_1] = [x_4, B^2 x_1] = [x_4, 0] = 0$$

and, analogously, $[x_0, x_0] = [x_0, x_1] = [x_0, x_2] = [x_1, x_1] = 0$. If $x_2 \in M$ then

$$0 = [x_1, x_2] = [Bx_2, x_2] = [Ax_2, x_2],$$

which by (4.1) implies that $x_2 \in \ker A \cap M \subseteq \ker B$ which is a contradiction to $Bx_2 = x_1 \neq 0$. Hence, $x_2 \notin M$ and there exists $\alpha \in \mathbb{C}$ such that $x_1 - \alpha x_2 \in M$ and

$$0 = [x_0 - \alpha x_1, x_1 - \alpha x_2] = [B(x_1 - \alpha x_2), x_1 - \alpha x_2] = [A(x_1 - \alpha x_2), x_1 - \alpha x_2].$$

Again (4.1) implies $x_1 - \alpha x_2 \in \ker A \cap M \subseteq \ker B$ in contradiction to $B(x_1 - \alpha x_2) = x_0 - \alpha x_1 \neq 0$ and (ii) follows. □

Remark 4.2. In the situation of Lemma 4.1 we also see by Theorem 3.1 that B has no two (linearly independent) Jordan chains of length 3 at 0. Moreover, by Corollary 3.3(ii) we have $\dim \mathfrak{L}_0(B) < \infty$ if $\dim \mathfrak{L}_0(A) < \infty$.

The existence of a long Jordan chain (of length 3 or 4) of B at 0 has consequences for the Jordan structure of B at 0. Statement (i) of Proposition 4.3 below is from [17].

Proposition 4.3. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that* (2.5) *holds for some real number $\lambda_0 \in \rho(A) \cap \rho(B) \cap \mathbb{R}$. Assume that A is non-negative and* 0 *is an isolated eigenvalue of A with $\dim \mathfrak{L}_0(A) < \infty$. Let B have a Jordan chain at* 0 *of length* 3*. Then the following holds.*

(i) $\ker B \subseteq \ker A$.

(ii) $\dim \mathfrak{L}_0(A) \geq \dim \ker B^2$.

(iii) $\dim \big(\ker A^2/\ker A\big) \leq \dim \big(\ker B^2/\ker B\big)$.

Proof. (i) Assume that $\{x_0, x_1, x_2\}$ is a Jordan chain of B at 0 of length 3 and let M be given as in (3.19). Let $y \in \ker B$ and assume $y \notin \ker A$. Then $y \notin M$ and there exists $\alpha \in \mathbb{C}$ such that $x_1 - \alpha y \in M$ and

$$\begin{aligned}[A(x_1 - \alpha y), x_1 - \alpha y] &= [B(x_1 - \alpha y), x_1 - \alpha y] = [x_0, x_1 - \alpha y] \\ &= -[x_0, \alpha y] = -[Bx_1, \alpha y] = 0.\end{aligned}$$

Here we used that $[x_0, x_1] = [B^2 x_2, x_1] = [x_2, B^2 x_1] = 0$. From (4.1) we then conclude $x_1 - \alpha y \in \ker A \cap M \subseteq \ker B$, but $B(x_1 - \alpha y) = x_0 \neq 0$, a contradiction. Thus we have $\ker B \subseteq \ker A$.

(ii) Let $\{x_0, x_1\}$ be a Jordan chain of B at 0 of length 2. Then x_0 is neutral, cf. Lemma 1.8. In particular, there exists a $\dim(\ker B^2/\ker B)$-dimensional neutral subspace N_0 in $\ker B$,

$$N_0 := \{Bx \mid x \in \ker B^2\}.$$

For $y_0 \in \ker B$, we have $[x_0, y_0] = [x_1, By_0] = 0$, hence $x_0 \in \ker B^{[\perp]}$. Let N be a $\big(\dim \ker B - \dim(\ker B^2/\ker B)\big)$-dimensional subspace of $\ker B$, such that

$$\ker B = N \dotplus N_0.$$

As seen above, we have $N[\perp]N_0$. Let $N = N_+[\dotplus]N_-[\dotplus]N^\circ$ be a decomposition of N into a positive subspace N_+, a negative subspace N_- and the isotropic part N° as in

Proposition 1.3. Summing up, we have

$$\ker B = N_+[\dot{+}]N_-[\dot{+}]N^\circ[\dot{+}]N_0.$$

Then $N_+[\dot{+}]N^\circ[\dot{+}]N_0$ ($N_-[\dot{+}]N^\circ[\dot{+}]N_0$) is a $(\dim N_+ + \dim N^\circ + \dim N_0)$-dimensional non-negative subspace (a $(\dim N_- + \dim N^\circ + \dim N_0)$-dimensional non-positive subspace, respectively) of $\ker B \subseteq \ker A \subseteq \mathfrak{L}_0(A)$. Let

$$\mathfrak{L}_0(A) = \mathfrak{L}_+[\dot{+}]\mathfrak{L}_-$$

be a Krein space decomposition of $\mathfrak{L}_0(A)$, see Proposition 1.12. With Proposition 1.4 it follows

$$\begin{aligned}
\dim \mathfrak{L}_0(A) &= \dim \mathfrak{L}_+ + \dim \mathfrak{L}_- \\
&\geq \dim N_+ + \dim N^\circ + \dim N_0 + \dim N_- + \dim N^\circ + \dim N_0 \\
&= \dim \ker B + \dim N_0 + \dim N^\circ \\
&= \dim \ker B + \dim(\ker B^2 / \ker B) + \dim N^\circ \\
&\geq \dim \ker B^2.
\end{aligned}$$

(iii) Let $\{x_0, x_1, x_2\}$ be a Jordan chain of B at 0 of length 3 and let M be given as in (3.19). According to (i) we have $x_0 \in \ker A \cap M$. Furthermore, $x_1 \notin M$. Indeed, if $x_1 \in M$, then $[Ax_1, x_1] = [Bx_1, x_1] = [x_0, x_1] = [x_0, Bx_2] = 0$, hence $x_1 \in \ker A$ and $0 = Ax_1 = Bx_1 = x_0$, a contradiction.

Assume $\dim(\ker A^2/\ker A) > \dim(\ker B^2/\ker B) =: n+1$. Then, there exist linearly independent vectors $\widetilde{y}_1, \dots, \widetilde{y}_n$, such that

$$\ker B^2 = \ker B \dotplus \operatorname{span}\{\widetilde{y}_1, \dots, \widetilde{y}_n, x_1\}.$$

As $x_1 \notin M$, we find α_j with $y_j := \widetilde{y}_j - \alpha_j x_1 \in M$, $j = 1, \dots, n$, and hence

$$\ker B^2 = \ker B \dotplus \operatorname{span}\{y_1, \dots, y_n, x_1\}.$$

Let z_1, z_2 be such that

$$\ker A^2 = \ker A \dotplus \operatorname{span}\{y_1, \dots, y_n, z_1, z_2\},$$

cf. Theorem 3.1. As $\ker B \subseteq \ker A$, $\{By_j, y_j\}$ are Jordan chains of A at 0 of length 2, $j = 1, \dots, n$. Observe that $\ker B \subseteq \ker A$ implies $\ker B \subseteq M$.

If $z_1, z_2 \in M$, then $Bz_i = Az_i$, $i = 1, 2$. So, if $Bz_i \in M$, then $\{Bz_i, z_i\}$ are Jordan chains of B at 0, $i = 1, 2$, in contradiction to $\dim(\ker B^2/\ker B) < \dim(\ker A^2/\ker A)$. Therefore, it is no restriction to assume $Bz_2 \notin M$. Then there exists $\alpha \in \mathbb{C}$ with $Bz_1 - \alpha Bz_2 \in M$. Hence $\{B(z_1 - \alpha z_2), z_1 - \alpha z_2\}$ is a Jordan chain of B at 0. As $\ker B \subseteq \ker A$ we see that $z_1 - \alpha z_2$ is linearly independent to $\ker B \dotplus \operatorname{span}\{y_1, \dots, y_n\}$ and hence

$$\ker B^2 = \ker B \dotplus \operatorname{span}\{y_1, \dots, y_n, z_1 - \alpha z_2\} \subseteq M, \tag{4.2}$$

in contradiction to $x_1 \in (\ker B^2)\backslash M$.

Hence z_1 or z_2 is not in M and it is no restriction to assume $z_2 \notin M$. Then there exists α with $z_1 - \alpha z_2 \in M$ and in particular $B(z_1 - \alpha z_2) = A(z_1 - \alpha z_2)$. If $B(z_1 - \alpha z_2) \in M$, then $\{B(z_1 - \alpha z_2), z_1 - \alpha z_2\}$ is again a Jordan chain of B at 0. As above (4.2) holds and we get a contradiction to $x_1 \in (\ker B^2)\backslash M$. Consequently, $B(z_1 - \alpha z_2) = A(z_1 - \alpha z_2) \notin M$. Moreover, $w := A(z_1 - \alpha z_2) \in (\ker A)\backslash M$. Thus we find γ, δ such that $v_1 := z_1 - \gamma w, v_2 := z_2 - \delta w \in M$ and we have

$$\ker A^2 = \ker A \dotplus \operatorname{span}\{y_1, \ldots, y_n, v_1, v_2\}$$

with $v_1, v_2 \in M$. But we showed above that this is not possible. Hence, we have a contradiction which concludes the proof. $\square$

The next proposition provides a special basis of $\mathfrak{L}_0(B)$ if B has a long Jordan chain at 0, cf. [17].

Proposition 4.4. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that* (2.5) *holds for some real number $\lambda_0 \in \rho(A) \cap \rho(B) \cap \mathbb{R}$. Assume that A is non-negative and* 0 *is an isolated eigenvalue of A with $\dim \mathfrak{L}_0(A) < \infty$. Let M be given as in* (3.19)*. If $\{x_0, x_1, x_2\}$ is a Jordan chain of B at* 0 *of length* 3 *and B has no Jordan chain at* 0 *of length* 4*, then there exists a basis $\mathfrak{B}$ of $\mathcal{L}_0(B)$ containing $\{x_0, x_1, x_2\}$ with*

$$\mathfrak{B}\backslash\{x_1, x_2\} \subseteq \mathcal{L}_0(A) \cap M.$$

If B has a Jordan chain $\{x_0, x_1, x_2, x_3\}$ at 0 *of length* 4*, then there exists a basis $\mathfrak{B}$ of $\mathcal{L}_0(B)$ containing $\{x_0, x_1, x_2, x_3\}$ with*

$$\mathfrak{B}\backslash\{x_1, x_2, x_3\} \subseteq \mathcal{L}_0(A) \cap M.$$

In both cases, $x_1 \notin M$.

Proof. We consider the case that there is a Jordan chain $\{x_0, x_1, x_2\}$ of B at 0 of length 3 and none of length 4. In this case we have $[x_0, x_0] = [x_1, x_0] = 0$. By Proposition 4.3 we have $\ker B \subseteq \ker A$ which implies $\ker B \subseteq M$. In particular, $x_0 \in M$. If $x_1 \in M$ then $[Ax_1, x_1] = [Bx_1, x_1] = [x_0, x_1] = 0$. Hence, by (4.1) $x_1 \in \ker A \cap M \subseteq \ker B$, a contradiction. Consequently, $x_1 \notin M$.

As $\dim \mathfrak{L}_0(A) < \infty$ it follows from Lemma 4.1(ii) and Corollary 3.3(ii) that the dimension $\dim \mathfrak{L}_0(B)$ of the root subspace $\mathcal{L}_0(B)$ is finite as well. If $\mathcal{L}_0(B) = \operatorname{span}\{x_0, x_1, x_2\}$ then the claim follows since $\ker B \subseteq \ker A$. Hence, let $\{x_0, x_1, x_2, u_3, \ldots, u_n\}$ be a basis of $\mathcal{L}_0(B)$ for some $n \geq 3$. For $3 \leq k \leq n$ we define z_k in the following way: If $u_k \in \ker B$ then $u_k \in \ker A \cap M$ and we set $z_k := u_k$. If $u_k \notin \ker B$ we obtain by Remark 4.2 $u_k \in \ker B^2$ and we set $y_k := Bu_k \neq 0$. As $x_1 \notin M$ there exist $\alpha_k \in \mathbb{C}$ such that $z_k := u_k - \alpha_k x_1 \in M$, $3 \leq k \leq n$, and we have

$$Az_k = Bz_k = y_k - \alpha_k x_0 \in \ker B \subseteq \ker A \quad \text{and} \quad z_k \in \ker A^2 = \mathcal{L}_0(A), \quad 3 \leq k \leq n.$$

The elements $x_0, x_1, x_2, z_3, \dots, z_n$ are linearly independent. Moreover, $x_0, z_3, \dots, z_n \in M \cap \ker B^2$. Thus $\mathfrak{B} := \{x_0, x_1, x_2, z_3, \dots, z_n\}$ is a basis of $\mathcal{L}_0(B)$ with the desired properties.

The case of a Jordan chain at 0 of length 4 is proved analogously. $\square$

With the basis of Proposition 4.4 we can improve the result in Proposition 4.3(iii) as follows.

Proposition 4.5. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that* (2.5) *holds for some real number $\lambda_0 \in \rho(A) \cap \rho(B) \cap \mathbb{R}$. Assume that A is non-negative and* 0 *is an isolated eigenvalue of A with* $\dim \mathfrak{L}_0(A) < \infty$. *Let B have a Jordan chain of length* ≥ 3 *at* 0 *and assume* $\dim \big(\ker A^2 / \ker A\big) = \dim \big(\ker B^2 / \ker B\big)$. *Then* $\ker A = \ker B$.

Proof. Let C and M be given as in (3.18) and (3.19), respectively. By Proposition 4.3(i) we have $\ker B \subseteq \ker A$. Assume $\dim \ker B = \dim \ker A - 1$. Since B has a Jordan chain $\{x_0, x_1, x_2\}$ ($\{x_0, x_1, x_2, x_3\}$) of length 3 (4, respectively), by Proposition 4.4 there exists a basis $\mathfrak{B}$ of $\mathfrak{L}_0(B)$, such that $\mathfrak{B} \backslash \{x_1, x_2\} \subseteq \mathfrak{L}_0(A) \cap M$ ($\mathfrak{B} \backslash \{x_1, x_2, x_3\} \subseteq \mathfrak{L}_0(A) \cap M$, respectively). Let $z_2, \dots, z_n \in \mathfrak{B} \cap (\ker B^2 \backslash \ker B)$ be a maximal number of vectors, such that $x_1, z_2, \dots, z_n$ are linearly independent. Then, $z_2, \dots, z_n \in M$. Setting $y_i := B z_i$, $i = 2, \dots, n$, the vectors $x_0, y_2, \dots, y_n \in \ker B \subseteq \ker A$ are as well linearly independent. We complete $\{x_0, y_2, \dots, y_n\}$ to a basis $\{x_0, y_2, \dots, y_m\}$ of $\ker B$, $m \geq n$. Moreover, let $y_{m+1}, z_{n+1} \in \mathcal{K}$, such that $\ker A = \operatorname{span}\{x_0, y_2 \dots, y_m, y_{m+1}\}$ and $\ker A^2 = \mathfrak{L}_0(A) = \operatorname{span}\{x_0, y_2 \dots, y_{m+1}, z_2, \dots, z_n, z_{n+1}\}$. Then $y_{m+1} \notin M$, because else $0 = A y_{m+1} = B y_{m+1}$, which gives $y_{m+1} \in \ker B$, in contradiction to $\dim \ker B = \dim \ker A - 1$. Hence, there exists $\eta \in \mathbb{C}$ such that $v := z_{n+1} - \eta y_{m+1} \in M$. Moreover, by Proposition 4.4 we have $x_1 \notin M$. There exist $\mu_0, \mu_2, \dots, \mu_{m+1} \in \mathbb{C}$ with

$$Av = A z_{n+1} = \mu_0 x_0 + \sum_{i=2}^{m} \mu_i y_i + \mu_{m+1} y_{m+1}.$$

Setting $u := \mu_0 x_0 + \sum\limits_{i=2}^{m} \mu_i y_i \in \ker B$ we thus have

$$Av = u + \mu_{m+1} y_{m+1} \tag{4.3}$$

and $\mu_{m+1} \neq 0$. Indeed, if $\mu_{m+1} = 0$ then $Av = u \in M$ implies $0 = A^2 v = BAv = B^2 v$. Hence, $v \in \ker B^2$, in contradiction to the linear independence of $\{v, x_1, z_2, \dots, z_n\}$. For every $b \in \ker B$ we have $[b, x_0] = [b, Bx_1] = [Bb, x_1] = 0$ and (4.3) gives $y_{m+1} = \frac{1}{\mu_{m+1}}(Av - u)$. This yields

$$[y_{m+1}, x_0] = \frac{1}{\mu_{m+1}} [Av - u, x_0] = \frac{1}{\mu_{m+1}} [v, Ax_0] = 0. \tag{4.4}$$

As $x_1 \notin M$, we see $(B - \lambda_0) x_1 \notin \ker C$, and therefore there exists $\beta \neq 0$, such that

$$\begin{aligned} 0 \neq \beta \varphi_A = C(B - \lambda_0) x_1 &= (B - \lambda_0)^{-1} (B - \lambda_0) x_1 - (A - \lambda_0)^{-1} (B - \lambda_0) x_1 \\ &= x_1 - (A - \lambda_0)^{-1} (x_0 - \lambda_0 x_1). \end{aligned}$$

Consequently, $(A-\lambda_0)^{-1}(x_0-\lambda_0 x_1) = x_1 - \beta\varphi_A \in \operatorname{dom} A$ and

$$x_0 - \lambda_0 x_1 = (A-\lambda_0)(x_1-\beta\varphi_A) = A(x_1-\beta\varphi_A) - \lambda_0 x_1 + \lambda_0\beta\varphi_A,$$

which is equivalent to $A(x_1-\beta\varphi_A) = x_0 - \lambda_0\beta\varphi_A$. With (4.4) we see

$$0 = [x_1-\beta\varphi_A, Ay_{m+1}] = [A(x_1-\beta\varphi_A), y_{m+1}] = [x_0-\lambda_0\beta\varphi_A, y_{m+1}] = -\lambda_0\beta[\varphi_A, y_{m+1}],$$

which implies $[\varphi_A, y_{m+1}] = 0$. This gives

$$[\varphi_A, (A-\lambda_0)y_{m+1}] = [\varphi_A, -\lambda_0 y_{m+1}] = 0,$$

i.e. $y_{m+1} \in M$ by (3.20), a contradiction. □

The following theorem is one of the main results of this chapter, cf. also [17]. It states that the dimension of the root subspaces of the unperturbed operator A and the perturbed operator B can only differ by 2. It plays an important role in Section 5.3.

Theorem 4.6. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that (2.5) holds for some real number $\lambda_0 \in \rho(A)\cap\rho(B)\cap\mathbb{R}$. Assume that A is non-negative and 0 is an isolated eigenvalue of A with $\dim \mathfrak{L}_0(A) < \infty$. Then*

$$|\dim \mathfrak{L}_0(A) - \dim \mathfrak{L}_0(B)| \leq 2$$

and this estimate is sharp.

Proof. By Lemma 4.1(i) and (ii) we see $\mathcal{L}_0(B) = \ker B^4$, $\mathcal{L}_0(A) = \ker A^2$, and with Theorem 3.1(i) we obtain

$$\dim \mathfrak{L}_0(A) - 2 = \dim\ker A^2 - 2 \leq \dim\ker B^2 \leq \dim\ker B^4 = \dim \mathfrak{L}_0(B). \tag{4.5}$$

By Corollary 3.3 the root subspace $\mathcal{L}_0(B)$ is finite dimensional. In regard of (4.5) it remains to prove

$$\dim \mathfrak{L}_0(B) \leq \dim \mathfrak{L}_0(A) + 2. \tag{4.6}$$

By Remark 4.2 B has at most a single Jordan chain of length 3 or 4 at 0. Hence, if $\dim\ker B^2 \leq \dim\ker A^2$ the claim follows. Therefore, assume $\dim\ker B^2 > \dim\ker A^2 = \dim \mathfrak{L}_0(A)$. If there is no Jordan chain of B at 0 of length 3 the estimate follows from Theorem 3.1(i). Now assume that B has a Jordan chain $\{x_0, x_1, x_2\}$ of length 3 at 0. Then we have by Proposition 4.3(ii) $\dim\ker B^2 \leq \dim \mathfrak{L}_0(A)$, a contradiction.

It remains to show the sharpness of (4.6). For this consider the space $\mathbb{C}^2$ with a fundamental symmetry J and operators A and B defined via

$$J = A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easily seen that A and B satisfy the assumptions of Theorem 4.6, $\dim \mathfrak{L}_0(A) = 0$, and $\dim \mathfrak{L}_0(B) = 2$. □

At the end of this section, we describe the Jordan structure of B at 0, i.e. the number and length of the linearly independent Jordan chains of B at 0 compared with the Jordan chains of A at 0. In view of Theorem 3.1(ii), Lemma 4.1, Proposition 4.3(i), and Theorem 4.6, we see that there are only 20 combinatorial possibilities for the structure of $\mathfrak{L}_0(B)$. But not all of these can occur, as we show below.

For brevity, we denote

$$\begin{aligned} \beta_1 &:= \dim \ker B, & \alpha_1 &:= \dim \ker A, \\ \beta_2 &:= \dim(\ker B^2 / \ker B), & \alpha_2 &:= \dim(\ker A^2 / \ker A), \\ \beta_3 &:= \dim(\ker B^3 / \ker B^2), \\ \beta_4 &:= \dim(\ker B^4 / \ker B^3). \end{aligned}$$

Then the Jordan structures of A and B at 0 are determined by these numbers.

Theorem 4.7. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that* (2.5) *holds for some real number $\lambda_0 \in \rho(A) \cap \rho(B) \cap \mathbb{R}$. Assume that A is non-negative and* 0 *is an isolated eigenvalue of A with $\dim \mathfrak{L}_0(A) < \infty$. Suppose the Jordan structure of A at* 0 *is given by α_1 and α_2 as above. If $\kappa_B = 0$ there are* 7 *possible Jordan structures of B at* 0 *and if $\kappa_B = 1$ there are* 11 *possible structures. These numbers are sharp in the sense that each of these cases is attained.*

We illustrate the cases with some exemplary figures, so-called Ferrer diagrams of the root subspaces, cf. for example [93].

$\mathfrak{L}_0(A)$

The adjacent picture represents a basis of $\mathfrak{L}_0(A)$: The first line represents a basis of $\ker A$ with α_1 elements and the second line represents the α_2 linearly independent vectors in $\ker A^2 \setminus \ker A$. Moreover, the columns of this picture represent the linearly independent Jordan chains of A at 0. Note that these figures are just for illustrations and the proof makes no use of them.

Proof of Theorem 4.7. We know $|\alpha_1 - \beta_1| \leq 1$ and $|\alpha_2 - \beta_2| \leq 1$ by Theorem 3.1. Moreover, $|\dim \mathfrak{L}_0(A) - \dim \mathfrak{L}_0(B)| \leq 2$ by Theorem 4.6 and $\beta_3, \beta_4 \in \{0, 1\}$ with $\beta_3 \geq \beta_4$ by Lemma 4.1. Together with Proposition 4.3(i), this leaves the 20 combinatorial possibilities given below. In each case we either give an example to show that this case is attained or a proof that this case is not possible. In the following, J denotes a fundamental symmetry in $\mathbb{C}^n$, where n is chosen accordingly.

	$\mathfrak{L}_0(A)$	In the following examples A is always chosen in such a way that its root subspace at 0 has the adjacent structure. In most cases A will be in $\mathbb{C}^{6\times 6}$. Note that in some cases A has to be in $\mathbb{C}^{8\times 8}$ whereas in other cases $A \in \mathbb{C}^{2\times 2}$ or $A \in \mathbb{C}^{4\times 4}$ would suffice.
	$\bullet\ \bullet\ \bullet\ \bullet$ $\bullet\ \bullet$	
Case	$\mathfrak{L}_0(B)$	**Example or Proof**
1. $\beta_1 = \alpha_1 - 1$ $\beta_2 = \alpha_2 - 1$ $\beta_3 = 0$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet$ $\bullet$	$A = \begin{pmatrix} 0&0&&&&\\ 1&0&&&&\\ &&0&&&\\ &&&0&&\\ &&&&0&0\\ &&&&1&0 \end{pmatrix},\ B = \begin{pmatrix} 0&0&&&&\\ 1&0&&&&\\ &&1&1&0&1\\ &&1&1&0&1\\ &&1&1&0&1\\ &&0&0&1&0 \end{pmatrix}\ (\kappa_B = 0),$ $J = \begin{pmatrix} 0&1&&&&\\ 1&0&&&&\\ &&1&&&\\ &&&1&&\\ &&&&0&1\\ &&&&1&0 \end{pmatrix},\ B - A = \begin{pmatrix} 0&&&&&\\ &0&&&&\\ &&1&1&0&1\\ &&1&1&0&1\\ &&1&1&0&1\\ &&0&0&0&0 \end{pmatrix}$
2. $\beta_1 = \alpha_1 - 1$ $\beta_2 = \alpha_2 - 1$ $\beta_3 = 1$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet$ $\bullet$ $\bullet$	Not possible: By Proposition 4.3(iii) we have $\dim(\ker A^2 / \ker A) \leq \dim(\ker B^2 / \ker B)$.
3. $\beta_1 = \alpha_1 - 1$ $\beta_2 = \alpha_2 - 1$ $\beta_3 = 1$ $\beta_4 = 1$	$\bullet\ \bullet\ \bullet$ $\bullet$ $\bullet$ $\bullet$	Not possible, see 2.
4. $\beta_1 = \alpha_1 - 1$ $\beta_2 = \alpha_2$ $\beta_3 = 0$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet$ $\bullet\ \bullet$	$A = \begin{pmatrix} 0&1&&&&\\ 0&0&&&&\\ &&0&1&&\\ &&0&0&&\\ &&&&0&\\ &&&&&0 \end{pmatrix},\ B = \begin{pmatrix} 0&1&&&&\\ 0&0&&&&\\ &&0&1&&\\ &&0&0&&\\ &&&&0&\\ &&&&&1 \end{pmatrix}\ (\kappa_B = 0),$ $J = \begin{pmatrix} 0&1&&&&\\ 1&0&&&&\\ &&0&1&&\\ &&1&0&&\\ &&&&1&\\ &&&&&1 \end{pmatrix},\ B - A = \begin{pmatrix} 0&&&&&\\ &0&&&&\\ &&0&&&\\ &&&0&&\\ &&&&0&\\ &&&&&1 \end{pmatrix}$
5. $\beta_1 = \alpha_1 - 1$ $\beta_2 = \alpha_2$ $\beta_3 = 1$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet$ $\bullet\ \bullet$ $\bullet$	Not possible: As $\dim(\ker A^2 / \ker A) = \dim(\ker B^2 / \ker B)$, Proposition 4.5 gives $\dim \ker A = \dim \ker B$.
6. $\beta_1 = \alpha_1 - 1$ $\beta_2 = \alpha_2$ $\beta_3 = 1$ $\beta_4 = 1$	$\bullet\ \bullet\ \bullet$ $\bullet\ \bullet$ $\bullet$ $\bullet$	Not possible, see 5.
7. $\beta_1 = \alpha_1 - 1$ $\beta_2 = \alpha_2 + 1$ $\beta_3 = 0$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet$ $\bullet\ \bullet\ \bullet$	$A = \begin{pmatrix} 0&1&&&&\\ 0&0&&&&\\ &&0&1&&\\ &&0&0&&\\ &&&&0&\\ &&&&&0 \end{pmatrix},\ B = \begin{pmatrix} 0&1&&&&\\ 0&0&&&&\\ &&0&1&&\\ &&0&0&&\\ &&&&0&1\\ &&&&0&0 \end{pmatrix}\ (\kappa_B = 0),$ $J = \begin{pmatrix} 0&1&&&&\\ 1&0&&&&\\ &&0&1&&\\ &&1&0&&\\ &&&&0&1\\ &&&&1&0 \end{pmatrix},\ B - A = \begin{pmatrix} 0&&&&&\\ &0&&&&\\ &&0&&&\\ &&&0&&\\ &&&&0&1\\ &&&&0&0 \end{pmatrix}$

Case	$\mathfrak{L}_0(B)$	Example or Proof
8. $\beta_1 = \alpha_1 - 1$ $\beta_2 = \alpha_2 + 1$ $\beta_3 = 1$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet$ $\bullet\ \bullet\ \bullet$ $\bullet$	$A = \begin{pmatrix} 0&1&&&&& \\ 0&0&&&&& \\ &&0&1&&& \\ &&0&0&&& \\ &&&&0&& \\ &&&&&0& \\ &&&&&&1 \end{pmatrix}$, $B = \begin{pmatrix} 0&1&&&&& \\ 0&0&&&&& \\ &&0&1&&& \\ &&0&0&&& \\ &&&&0&0&0 \\ &&&&-1&0&-1 \\ &&&&-1&0&0 \end{pmatrix}$ $(\kappa_B = 1)$, $J = \begin{pmatrix} 0&1&&&&& \\ 1&0&&&&& \\ &&0&1&&& \\ &&1&0&&& \\ &&&&0&1& \\ &&&&1&0& \\ &&&&&&1 \end{pmatrix}$, $B-A = \begin{pmatrix} 0&&&&&& \\ &0&&&&& \\ &&0&&&& \\ &&&0&&& \\ &&&&0&0&0 \\ &&&&-1&0&-1 \\ &&&&-1&0&-1 \end{pmatrix}$
9. $\beta_1 = \alpha_1 - 1$ $\beta_2 = \alpha_2 + 1$ $\beta_3 = 1$ $\beta_4 = 1$	$\bullet\ \bullet\ \bullet$ $\bullet\ \bullet\ \bullet$ $\bullet$ $\bullet$	$A = \begin{pmatrix} 0&1&&&&&& \\ 0&0&&&&&& \\ &&0&1&&&& \\ &&0&0&&&& \\ &&&&0&0.5&0&0.5 \\ &&&&0&0&1&0 \\ &&&&0&0.5&0&0.5 \\ &&&&0&0&0&0 \end{pmatrix}$, $B = \begin{pmatrix} 0&1&&&&&& \\ 1&0&&&&&& \\ &&0&1&&&& \\ &&0&0&&&& \\ &&&&0&1&0&0 \\ &&&&0&0&1&0 \\ &&&&0&0&0&1 \\ &&&&0&0&0&0 \end{pmatrix}$ $(\kappa_B = 1)$, $J = \begin{pmatrix} 0&1&&&&&& \\ 1&0&&&&&& \\ &&0&1&&&& \\ &&1&0&&&& \\ &&&&0&0&0&1 \\ &&&&0&0&1&0 \\ &&&&0&1&0&0 \\ &&&&1&0&0&0 \end{pmatrix}$, $B-A = \begin{pmatrix} 0&&&&&&& \\ &0&&&&&& \\ &&0&&&&& \\ &&&0&&&& \\ &&&&0&0.5&0&-0.5 \\ &&&&0&0&0&0 \\ &&&&0&-0.5&0&0.5 \\ &&&&0&0&0&0 \end{pmatrix}$
10. $\beta_1 = \alpha_1$ $\beta_2 = \alpha_2 - 1$ $\beta_3 = 0$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet\ \bullet$ $\bullet$	Not possible: Since $\dim\ker A = \dim\ker B$, we see from Proposition 3.10(ii), that $\dim \mathfrak{L}_0(A) = \dim \mathfrak{L}_0(B)$.
11. $\beta_1 = \alpha_1$ $\beta_2 = \alpha_2 - 1$ $\beta_3 = 1$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet\ \bullet$ $\bullet$ $\bullet$	Not possible, see 2.
12. $\beta_1 = \alpha_1$ $\beta_2 = \alpha_2 - 1$ $\beta_3 = 1$ $\beta_4 = 1$	$\bullet\ \bullet\ \bullet\ \bullet$ $\bullet$ $\bullet$ $\bullet$	Not possible, see 2.
13. $\beta_1 = \alpha_1$ $\beta_2 = \alpha_2$ $\beta_3 = 0$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet\ \bullet$ $\bullet\ \bullet$	$A = \begin{pmatrix} 0&1&&&&& \\ 0&0&&&&& \\ &&0&1&&& \\ &&0&0&&& \\ &&&&0&& \\ &&&&&0& \\ &&&&&&1 \end{pmatrix}$, $B = \begin{pmatrix} 0&1&&&&& \\ 0&0&&&&& \\ &&0&1&&& \\ &&0&0&&& \\ &&&&0&& \\ &&&&&0& \\ &&&&&&2 \end{pmatrix}$ $(\kappa_B = 0)$, $J = \begin{pmatrix} 0&1&&&&& \\ 1&0&&&&& \\ &&0&1&&& \\ &&1&0&&& \\ &&&&1&& \\ &&&&&1& \\ &&&&&&1 \end{pmatrix}$, $B-A = \begin{pmatrix} 0&&&&&& \\ &0&&&&& \\ &&0&&&& \\ &&&0&&& \\ &&&&0&& \\ &&&&&0& \\ &&&&&&1 \end{pmatrix}$

Case	$\mathfrak{L}_0(B)$	Example or Proof
14. $\beta_1 = \alpha_1$ $\beta_2 = \alpha_2$ $\beta_3 = 1$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet\ \bullet$ $\bullet\ \bullet$ $\bullet$	$A = \begin{pmatrix} 0 &&&&&& \\ & 0 &&&&& \\ && 0 & 1 &&& \\ && 0 & 0 &&& \\ &&&& 0 & 1 & \\ &&&& 0 & 0 & \\ &&&&&& 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 &&&&&& \\ & 0 &&&&& \\ && 0 & 1 &&& \\ && 0 & 0 &&& \\ &&&& 0 & 0 & -1 \\ &&&& 0 & 0 & 0 \\ &&&& 0 & -1 & 0 \end{pmatrix}$ $(\kappa_B = 1)$, $J = \begin{pmatrix} 1 &&&&&& \\ & 1 &&&&& \\ && 0 & 1 &&& \\ && 1 & 0 &&& \\ &&&& 0 & 1 & \\ &&&& 1 & 0 & \\ &&&&&& 1 \end{pmatrix}$, $B - A = \begin{pmatrix} 0 &&&&&& \\ & 0 &&&&& \\ && 0 &&&& \\ &&& 0 &&& \\ &&&& 0 & -1 & -1 \\ &&&& 0 & 0 & 0 \\ &&&& 0 & -1 & -1 \end{pmatrix}$
15. $\beta_1 = \alpha_1$ $\beta_2 = \alpha_2$ $\beta_3 = 1$ $\beta_4 = 1$	$\bullet\ \bullet\ \bullet\ \bullet$ $\bullet\ \bullet$ $\bullet$ $\bullet$	$A = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 & 1 &&&& \\ && 0 & 0 &&&& \\ &&&& 0 & 0 & 0 & 1 \\ &&&& 0 & 0 & 1 & 0 \\ &&&& 0 & 1 & 0 & 0 \\ &&&& 0 & 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 & 1 &&&& \\ && 0 & 0 &&&& \\ &&&& 0 & 1 & 0 & 0 \\ &&&& 0 & 0 & 1 & 0 \\ &&&& 0 & 0 & 0 & 1 \\ &&&& 0 & 0 & 0 & 0 \end{pmatrix}$ $(\kappa_B = 1)$, $J = \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 0 & 1 &&&& \\ && 1 & 0 &&&& \\ &&&& 0 & 0 & 0 & 1 \\ &&&& 0 & 0 & 1 & 0 \\ &&&& 0 & 1 & 0 & 0 \\ &&&& 1 & 0 & 0 & 0 \end{pmatrix}$, $B - A = \begin{pmatrix} 0 &&&&&&& \\ & 0 &&&&&& \\ && 0 &&&&& \\ &&& 0 &&&& \\ &&&& 0 & 1 & 0 & -1 \\ &&&& 0 & 0 & 0 & 0 \\ &&&& 0 & -1 & 0 & 1 \\ &&&& 0 & 0 & 0 & 0 \end{pmatrix}$
16. $\beta_1 = \alpha_1$ $\beta_2 = \alpha_2 + 1$ $\beta_3 = 0$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet\ \bullet$ $\bullet\ \bullet\ \bullet$	Not possible, see 10.
17. $\beta_1 = \alpha_1$ $\beta_2 = \alpha_2 + 1$ $\beta_3 = 1$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet\ \bullet$ $\bullet\ \bullet\ \bullet$ $\bullet$	Not possible: By Proposition 4.3(ii), we have $\dim \mathfrak{L}_0(A) \geq \dim \ker B^2$, a contradiction.
18. $\beta_1 = \alpha_1 + 1$ $\beta_2 = \alpha_2 - 1$ $\beta_3 = 0$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet\ \bullet\ \bullet$ $\bullet$	$A = \begin{pmatrix} 0 &&&&& \\ & 0 &&&& \\ && 0 & 1 && \\ && 0 & 0 && \\ &&&& 0 & 1 \\ &&&& 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 &&&&& \\ & 0 &&&& \\ && 0 & 1 && \\ && 0 & 0 && \\ &&&& 0 & \\ &&&&& 0 \end{pmatrix}$ $(\kappa_B = 0)$, $J = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 0 & 1 && \\ && 1 & 0 && \\ &&&& 0 & 1 \\ &&&& 1 & 0 \end{pmatrix}$, $B - A = \begin{pmatrix} 0 &&&&& \\ & 0 &&&& \\ && 0 &&& \\ &&& 0 && \\ &&&& 0 & -1 \\ &&&& 0 & 0 \end{pmatrix}$
19. $\beta_1 = \alpha_1 + 1$ $\beta_2 = \alpha_2$ $\beta_3 = 0$ $\beta_4 = 0$	$\bullet\ \bullet\ \bullet\ \bullet\ \bullet$ $\bullet\ \bullet$	$A = \begin{pmatrix} 0 & 1 &&&&& \\ 0 & 0 &&&&& \\ && 0 & 1 &&& \\ && 0 & 0 &&& \\ &&&& 0 && \\ &&&&& 0 & \\ &&&&&& 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 &&&&& \\ 0 & 0 &&&&& \\ && 0 & 1 &&& \\ && 0 & 0 &&& \\ &&&& 0 && \\ &&&&& 0 & \\ &&&&&& 0 \end{pmatrix}$ $(\kappa_B = 0)$, $J = \begin{pmatrix} 0 & 1 &&&&& \\ 1 & 0 &&&&& \\ && 0 & 1 &&& \\ && 1 & 0 &&& \\ &&&& 1 && \\ &&&&& 1 & \\ &&&&&& 1 \end{pmatrix}$, $B - A = \begin{pmatrix} 0 &&&&&& \\ & 0 &&&&& \\ && 0 &&&& \\ &&& 0 &&& \\ &&&& 0 && \\ &&&&& 0 & \\ &&&&&& -1 \end{pmatrix}$

Case		$\mathfrak{L}_0(B)$	Example or Proof
20.	$\beta_1 = \alpha_1 + 1$ $\beta_2 = \alpha_2 + 1$ $\beta_3 = 0$ $\beta_4 = 0$	• • • • • • • •	$A = \begin{pmatrix} 0&1&&&&&& \\ 0&0&&&&&& \\ &&0&0&&&& \\ &&1&0&&&& \\ &&&&1&1&0&1 \\ &&&&1&1&0&1 \\ &&&&1&1&0&1 \\ &&&&0&0&1&0 \end{pmatrix}$, $B = \begin{pmatrix} 0&1&&&&&& \\ 0&0&&&&&& \\ &&0&0&&&& \\ &&1&0&&&& \\ &&&&0&&& \\ &&&&&0&& \\ &&&&&&0&0 \\ &&&&&&1&0 \end{pmatrix}$, $(\kappa_B = 0)$ $J = \begin{pmatrix} 0&1&&&&&& \\ 1&0&&&&&& \\ &&0&1&&&& \\ &&1&0&&&& \\ &&&&1&&& \\ &&&&&1&& \\ &&&&&&0&1 \\ &&&&&&1&0 \end{pmatrix}$, $B - A = \begin{pmatrix} 0&&&&&&& \\ &0&&&&&& \\ &&0&&&&& \\ &&&0&&&& \\ &&&&-1&-1&0&-1 \\ &&&&-1&-1&0&-1 \\ &&&&-1&-1&0&-1 \\ &&&&0&0&0&0 \end{pmatrix}$

Table 4.1: Possible structures of $\mathfrak{L}_0(B)$.

Hence, we see that 11 cases can occur. If $\kappa_B = 0$, then the cases $8, 9, 14$ and 15 neither are possible, cf. Lemma 4.1, and in the remaining cases we indeed find examples with $\kappa_B = 0$. This completes the proof. $\square$

4.2 Root Subspaces at $\mu \neq 0$

We now describe the root subspaces of A and B at $\mu \neq 0$. For this, let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ such that (2.5) holds for some real number $\lambda_0 \in \rho(A) \cap \rho(B) \cap \mathbb{R}$ and assume that A is non-negative. The structure of the non-real spectrum of the perturbed operator B is very simple, as the following remark shows.

Remark 4.8. As B is a one-dimensional perturbation of A, by Theorem 1.13(iii) B is either non-negative or has one negative square. Consequently, B has at most one pair $\{\mu, \overline{\mu}\}$ of non-real eigenvalues with $\dim \mathfrak{L}_\mu(B) = \dim \mathfrak{L}_{\overline{\mu}}(B) = 1$, cf. Theorem 1.13(i).

The following example illustrates that non-real spectrum can occur.

Example 4.9. Consider the matrices

$$A = J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad C = B - A = \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}.$$

Then A is non-negative, $B - A$ has one-dimensional range and $\sigma(B) = \{i, -i\}$.

It remains to investigate non-zero real points in the spectrum. Let $\mu \neq 0$ is an isolated eigenvalue of A with finite algebraic multiplicity. Then we have by Proposition 1.12 the decomposition $\mathcal{K} = \mathfrak{L}_\mu(A)[\dot{+}]\mathfrak{L}_\mu(A)^{[\perp]}$. By Theorem 1.13(ii) the length of the Jordan chains of B at μ is bounded by 3. Hence $\dim \mathfrak{L}_\mu(B) < \infty$ by Corollary 3.3(ii). Moreover, we have the following bound on the change of the dimension of the root subspaces of A and B.

Corollary 4.10. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot, \cdot])$ such that (2.5) holds for some real number $\lambda_0 \in \rho(A) \cap \rho(B) \cap \mathbb{R}$. Assume that A is non-negative and*

$\mu \neq 0$ is an isolated eigenvalue of A with $\dim \mathfrak{L}_\mu(A) < \infty$. Then

$$\big|\dim \mathfrak{L}_\mu(A) - \dim \mathfrak{L}_\mu(B)\big| \leq 3 \tag{4.7}$$

and this estimate is sharp.

Proof. According to Theorem 1.13(ii), A and B have Jordan chains at $\mu \neq 0$ of length at most 1 and 3, respectively. Hence, (4.7) follows from Corollary 3.3(ii). It remains to show the sharpness of (4.7). For this consider the space $\mathbb{C}^3$ with the fundamental symmetry J and operators A and B defined via

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easily seen that A, B, and $\mu = 1$ satisfy the assumptions of Corollary 4.10, $m_A(\{1\}) = 0$, and $m_B(\{1\}) = 3$. □

Lemma 4.11. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that* (2.5) *holds for some real number $\lambda_0 \in \rho(A) \cap \rho(B) \cap \mathbb{R}$. Assume that A is non-negative and $\mu \neq 0$ is an isolated eigenvalue of A with $\dim \mathfrak{L}_\mu(A) < \infty$. If B has a Jordan chain at μ of length ≥ 2, then $\ker(A-\mu) \subseteq \ker(B-\mu)$ with $\dim\ker(A-\mu) = \dim\ker(B-\mu) - 1$.*

Proof. In what follows we show that

$$\text{(i)} \dim\ker(A-\mu) = \dim\ker(B-\mu) + 1 \quad \text{and} \quad \text{(ii)} \dim\ker(A-\mu) = \dim\ker(B-\mu)$$

are not possible. For this let $\mu \in (0,\infty)$. The proof for $\mu \in (-\infty,0)$ is analogously. Denote by $\{x_0, x_1\}$ a Jordan chain of B at μ and let M be given as in (3.19). Then x_0 is neutral by Lemma 1.8 and hence $x_0 \notin \ker(A-\mu)$ by Corollary 1.14(i). But as $x_0 \in \ker(B-\mu)$, we see $x_0 \notin M$. Let $\{x_0, \widetilde{y}_1, \ldots, \widetilde{y}_n\}$ be a basis of $\ker(B-\mu)$. Then there exist α_j such that $y_j := \widetilde{y}_j - \alpha_j x_0 \in M$, $j = 1, \ldots, n$, and $\{x_0, y_1, \ldots, y_n\}$ is a basis of $\ker(B-\mu)$ with $\{y_1, \ldots, y_n\} \subseteq \ker(A-\mu)$.

(i) Assume $\dim\ker(A-\mu) = \dim\ker(B-\mu) + 1$ and let $\{y_1, \ldots, y_n, z_1, z_2\}$ be a basis of $\ker(A-\mu)$. If $z_1, z_2 \in M$, then $z_1, z_2 \in \ker(B-\mu)$, a contradiction. Hence, we can assume $z_2 \notin M$. Then $z_1 - \alpha z_2 \in M$ for a suitable α, so that $z_1 - \alpha z_2 \in \ker(B-\mu)$ admits the representation

$$z_1 - \alpha z_2 = \sum_{j=1}^{n} \beta_j y_j + \gamma x_0,$$

which gives

$$\gamma x_0 = z_1 - \alpha z_2 - \sum_{j=1}^{n} \beta_j y_j \in M,$$

and hence $\gamma = 0$. Thus, $z_1 - \alpha z_2$ is a linear combination of $y_1, \ldots, y_n$, a contradiction.

(ii) As $\mu \in \sigma(A)$ is isolated we find a bounded interval I such that $I \cap \sigma(A) = \{\mu\}$. We see by Proposition 4.1 that μ is a pole of the resolvent of A. Since $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B)$ by Weyl's essential spectrum theorem, we have $\rho(B) \cap I \neq \emptyset$. Hence A, B, and I fulfil Assumption *I*. By Corollary 1.14 we have $\mu \in \sigma_{++}(A)$ and hence $(\ker(A-\mu), [\cdot,\cdot])$ is a Hilbert space. Choose $y_1, \dots, y_n$ and $z \in \{y_1, \dots, y_n\}^{[\perp]}$ such that $\{y_1, \dots, y_n, z\}$ is a basis of $\ker(A-\mu)$. We set $E = \operatorname{span}\{y_1, \dots, y_n\}$, then

$$A = \begin{pmatrix} \mu I & 0 \\ 0 & A_0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \mu I & 0 \\ 0 & B_0 \end{pmatrix}$$

holds with respect to $\mathcal{K} = E[\dot{+}]E^{[\perp]}$, where A_0 is non-negative, $(A_0 - \lambda)^{-1} - (B_0 - \lambda)^{-1}$ is of rank one, and A_0, B_0 satisfy the assumptions of the lemma. As $A_0 z = \mu z$, we have $\mu \in \sigma_{++}(A_0)$. Let P be the selfadjoint projection onto E (with respect to $[\cdot,\cdot]$), cf. Proposition 1.9. Then $0 = [x_0, x_0] = [Px_0, Px_0] + [(I-P)x_0, (I-P)x_0]$. With $[Px_0, Px_0] > 0$ if $Px_0 \neq 0$, we obtain $[(I-P)x_0, (I-P)x_0] \leq 0$ and $(I-P)x_0 \neq 0$ since $x_0 \notin E$. We have $B(I-P)x_0 = B_0(I-P)x_0 = \mu(I-P)x_0$, hence $\mu \in \sigma_{\mathrm{p}}(B_0)$. If $\mu \in \sigma_{\mathrm{p}}(S_0)$, $S_0 := A_0 \cap B_0$, then there exists $x \in M$ such that $x \in \ker(A_0 - \mu) \cap \ker(B_0 - \mu)$. But $\dim\ker(A_0 - \mu) = \dim\ker(B_0 - \mu) = 1$, hence $x \in \operatorname{span}\{z\}$, which gives $[x,x] > 0$, and $x \in \operatorname{span}\{(I-P)x_0\}$, which gives $[x,x] \leq 0$. Therefore $\mu \notin \sigma_{\mathrm{p}}(S_0)$, and by Proposition 2.16 $\mu \in \rho(B_0)$, in contradiction to above. In particular, $\ker(A-\mu) \subseteq \ker(B-\mu)$. $\square$

At the end of this section, we describe the Jordan structure of B at $\mu \neq 0$, i.e. the number and length of the linearly independent Jordan chains of B at μ compared with the Jordan chains of A at μ. In view of Theorem 1.13(ii), Theorem 3.1(ii) and Corollary 4.10, we see that there are only 9 combinatorial possibilities for the structure of $\mathfrak{L}_\mu(B)$. But not all of these can occur, as we show below.

For brevity, we denote

$$\begin{aligned} \beta_1 &:= \dim\ker(B-\mu), & \alpha_1 &:= \dim\ker(A-\mu), \\ \beta_2 &:= \dim(\ker(B-\mu)^2 / \ker(B-\mu)), & & \\ \beta_3 &:= \dim(\ker(B-\mu)^3 / \ker(B-\mu)^2). & & \end{aligned}$$

Then the Jordan structure of A and B at μ are determined by these numbers. The following result complements the results of Theorem 4.7.

Theorem 4.12. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that* (2.5) *holds for some real number $\lambda_0 \in \rho(A) \cap \rho(B) \cap \mathbb{R}$. Assume that A is non-negative and $\mu \neq 0$ is an isolated eigenvalue of A with $\dim \mathfrak{L}_\mu(A) < \infty$. Let $\alpha_1 = \dim\ker(A-\mu)$. If $\kappa_B = 0$ there are* 3 *possible Jordan structures of B at μ and if $\kappa_B = 1$ there are* 5 *possible structures. These numbers are sharp in the sense that each of these cases is attained.*

As in Theorem 4.7 we will illustrate the cases with Ferrer diagrams of the root subspaces. Note that these figures are just for illustrations and the proof makes no use of them.

$\mathfrak{L}_\mu(A)$

The adjacent picture represents the α_1 basis elements of $\ker(A-\mu) = \mathfrak{L}_\mu(A)$.

Proof of Theorem 4.12. We know $|\alpha_1 - \beta_1| \leq 1$ by Theorem 3.1. Moreover,

$$\left|\dim \mathfrak{L}_\mu(A) - \dim \mathfrak{L}_\mu(B)\right| \leq 3$$

by Corollary 4.10 and $\beta_2, \beta_3 \in \{0,1\}$ with $\beta_2 \geq \beta_3$ by Theorem 1.13. Together with Lemma 4.11(i), this leaves the 9 combinatorial possibilities given below. In each case we either give an example to show that this case is attained or a proof that this case is not possible. In the following, J denotes a fundamental symmetry in $\mathbb{C}^n$, where n is chosen accordingly.

	$\mathfrak{L}_\mu(A)$ • • •	In the following examples A is always chosen in such a way that its root subspace at μ has the adjacent structure. Note that in some cases a smaller matrix would suffice.
Case	$\mathfrak{L}_\mu(B)$	**Example or Proof**
1. $\beta_1 = \alpha_1 - 1$ $\beta_2 = 0$ $\beta_3 = 0$	• •	$\mu = 1$: $A = \mathrm{id},\ B = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}\ (\kappa_B = 0),\ J = \mathrm{id},\ B - A = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$
2. $\beta_1 = \alpha_1 - 1$ $\beta_2 = 1$ $\beta_3 = 0$	• • •	Not possible: By Lemma 4.11, we have $\dim\ker(A-\mu) = \dim\ker(B-\mu) - 1$.
3. $\beta_1 = \alpha_1 - 1$ $\beta_2 = 1$ $\beta_3 = 1$	• • • •	Not possible, see 2.
4. $\beta_1 = \alpha_1$ $\beta_2 = 0$ $\beta_3 = 0$	• • •	$\mu = 1$: $A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 2 \end{pmatrix} (\kappa_B = 0),\ J = \mathrm{id}, B - A = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 2 \end{pmatrix}$
5. $\beta_1 = \alpha_1$ $\beta_2 = 1$ $\beta_3 = 0$	• • • •	Not possible, see 2.
6. $\beta_1 = \alpha_1$ $\beta_2 = 1$ $\beta_3 = 1$	• • • • •	Not possible, see 2.
7. $\beta_1 = \alpha_1 + 1$ $\beta_2 = 0$ $\beta_3 = 0$	• • • •	$\mu = 1$: $A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 2 \end{pmatrix},\ B = \mathrm{id}\ (\kappa_B = 0),\ J = \mathrm{id},\ B - A = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}$
8. $\beta_1 = \alpha_1 + 1$ $\beta_2 = 1$ $\beta_3 = 0$	• • • • •	$\mu = 1$: $A = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & 1 \\ & & & 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & 1 \\ & & & 0 & 1 \end{pmatrix} (\kappa_B = 1),$ $J = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix},\ B - A = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & 0 \\ & & & -1 & 0 \end{pmatrix}$

Case	$\mathfrak{L}_\mu(B)$	Example or Proof
9. $\beta_1 = \alpha_1 + 1$ $\beta_2 = 1$ $\beta_3 = 1$		$\mu = 1$: $A = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 0 & 1 & 1 \\ &&& 0 & 1 & 1 \\ &&& 1 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 1 & 1 & 0 \\ &&& 0 & 1 & 1 \\ &&& 0 & 0 & 1 \end{pmatrix}$ $(\kappa_B = 1)$, $J = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 0 & 0 & 1 \\ &&& 0 & 1 & 0 \\ &&& 1 & 0 & 0 \end{pmatrix}$, $B - A = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 1 & 0 & -1 \\ &&& 0 & 0 & 0 \\ &&& -1 & 0 & 1 \end{pmatrix}$

Table 4.2: Possible structures of $\mathfrak{L}_\mu(B)$ for $\mu \in \mathbb{R}\backslash\{0\}$.

Hence, we see that 5 cases remain. If $\kappa_B = 0$, then the cases 8 and 9 neither are possible, cf. Corollary 1.14, and in the remaining cases we indeed find examples with $\kappa_B = 0$. This completes the proof. $\square$

5. Spectral Intervals under Rank One Perturbations

In this chapter we give some estimates on the change of the number of distinct eigenvalues and on the change of the total multiplicity of eigenvalues in an interval. As a vital instrument we use so-called $\mathcal{D}_0$- and $\mathcal{D}_1$-functions, which we introduce in Section 5.1. The results of Sections 5.2 and 5.3 are contained in [17].

5.1 $\mathcal{D}_0$- and $\mathcal{D}_1$-Functions

Spectral properties of selfadjoint operators in Hilbert and Krein spaces are closely connected to the properties of complex valued functions. For a selfadjoint operator A in some Hilbert space $\mathcal{H}$ and fixed $f \in \mathcal{H}$, the function

$$\left((A-\lambda)^{-1}f, f\right), \quad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

is a *Nevanlinna function.* Roughly speaking, the same relationship holds between *generalized Nevanlinna functions* and resolvents of selfadjoint operators in Pontryagin spaces and also between non-negative operators in Krein spaces and $\mathcal{D}_0$*-functions.* Moreover, the Weyl functions introduced in Chapter 2 turn out to be $\mathcal{D}_\kappa$*-functions*, see [24, Lemma 7].

The focus of this thesis lies on the investigation of non-negative operators and operators with one negative square. Therefore, we only discuss $\mathcal{D}_0$- and $\mathcal{D}_1$-functions, cf. [24, 25]. To this end we also introduce Nevanlinna functions and generalized Nevanlinna functions with one negative square, for more information see for example [69, 74, 77, 78, 79, 80].

We say, that $\lambda_0 \in \mathbb{R}$ is the *non-tangential limit from* $\mathbb{C}^+$ of the sequence $(\lambda_n) \subseteq \mathbb{C}^+$, if there exists $0 < \varepsilon < \frac{\pi}{2}$ such that for all $n \in \mathbb{N}$ we have $\varepsilon < \arg(\lambda_n - \lambda_0) < \pi - \varepsilon$. Likewise, ∞ is the *non-tangential limit from* $\mathbb{C}^+$ of $(\lambda_n) \subseteq \mathbb{C}^+$ ($\lambda_n \widehat{\to} \infty$), if there exists $0 < \varepsilon < \frac{\pi}{2}$ such that for all n we have $\varepsilon < \arg\lambda_n < \pi - \varepsilon$. If for a function f and $\lambda_0 \in \overline{\mathbb{R}}$ the limit $\lim_{n\to\infty} f(\lambda_n)$ exists and coincides for all sequences $(\lambda_n) \subseteq \mathbb{C}^+$ with $\lambda_n \widehat{\to} \lambda_0$, we call

$$\lim_{\lambda \widehat{\to} \lambda_0} f(\lambda) := \lim_{n\to\infty} f(\lambda_n)$$

the *non-tangential limit from* $\mathbb{C}^+$ *of* f *at* λ_0. The non-tangential limit from $\mathbb{C}^-$ is defined accordingly.

Let $M(\mathbb{C}\backslash\mathbb{R})$ denote the class of all functions τ which are meromorphic in both $\mathbb{C}^+$ and $\mathbb{C}^-$, and symmetric with respect to the real axis, i.e. $\tau(\overline{\lambda}) = \overline{\tau(\lambda)}$. If for some $\lambda_0 \in \overline{\mathbb{R}}$ the non-tangential limit $\lim_{\lambda \hat{\to} \lambda_0} \tau(\lambda)$ from $\mathbb{C}^+$ of τ exists and is real, we set $\tau(\lambda_0) := \lim_{\lambda \hat{\to} \lambda_0} \tau(\lambda)$. In this case, by the symmetry of τ, the non-tangential limit from $\mathbb{C}^-$ exists and has the same value. The union of all points of holomorphy of τ in $\mathbb{C}\backslash\mathbb{R}$ and all points $\lambda \in \overline{\mathbb{R}}$, such that τ can be analytically continued to λ and the continuations from $\mathbb{C}^+$ and $\mathbb{C}^-$ coincide, is denoted by $\mathfrak{h}(\tau)$. For the following definition see for example [25, Section 2].

Definition 5.1. Let $\tau \in M(\mathbb{C}\backslash\mathbb{R})$. If there exist constants $\mathcal{M}, m > 0$ and an open neighbourhood $\mathcal{U}$ of $\overline{\mathbb{R}}$ in $\overline{\mathbb{C}}$, such that $\mathcal{U}\backslash\overline{\mathbb{R}} \subseteq \mathfrak{h}(\tau)$ and

$$|\tau(\lambda)| \leq \frac{\mathcal{M}(1+|\lambda|)^{2m}}{|\operatorname{Im}\lambda|^m} \tag{5.1}$$

holds for all $\lambda \in \mathcal{U}\backslash\overline{\mathbb{R}}$ we will say that the *growth of τ near $\overline{\mathbb{R}}$ is of finite order.* An open subset $\Delta \subseteq \overline{\mathbb{R}}$ is said to be of *positive type with respect to* τ if for every sequence $(\lambda_n) \subseteq \mathfrak{h}(\tau) \cap \mathbb{C}^+$ which converges in $\overline{\mathbb{C}}$ to a point of Δ we have

$$\liminf_{n\to\infty} \operatorname{Im}\tau(\lambda_n) \geq 0.$$

An open subset $\Delta \subseteq \overline{\mathbb{R}}$ is said to be of *negative type with respect to* τ if Δ is of positive type with respect to $-\tau$.

Note, that open subsets $\Delta \subseteq \mathfrak{h}(\tau) \cap \overline{\mathbb{R}}$ are of positive and negative type with respect to τ. Let the growth of $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ near $\overline{\mathbb{R}}$ be of finite order. Let $\alpha \in \mathbb{R}$ and assume that there exists an open interval I_α, $\alpha \in I_\alpha$, such that $I_\alpha\backslash\{\alpha\}$ is of positive type with respect to τ. Let $\nu_\alpha \geq 0$ be the smallest integer such that

$$-\infty < \lim_{\lambda \hat{\to} \alpha} (\lambda-\alpha)^{2\nu_\alpha+1}\tau(\lambda) \leq 0. \tag{5.2}$$

Due to the finite-order growth of τ near $\overline{\mathbb{R}}$ such an integer ν_α always exists. To see this, let $(\lambda_n) \subseteq \mathbb{C}^+$, $\lambda_n \to \alpha$, and $0 < \varepsilon < \frac{\pi}{2}$ with $\varepsilon < \arg(\lambda_n - \alpha) < \pi - \varepsilon$ for all n. Then there exists $K > 0$ such that $\operatorname{Im}(\lambda_n - \alpha) \geq K|\operatorname{Re}(\lambda_n - \alpha)|$. With $\mathcal{M}, m > 0$ such that (5.1) holds we see for $2\nu_\alpha > m$

$$\begin{aligned}
|(\lambda_n-\alpha)^{2\nu_\alpha+1}\tau(\lambda_n)|^2 &\leq \Big(|\operatorname{Re}(\lambda_n-\alpha)|^2 + |\operatorname{Im}(\lambda_n-\alpha)|^2\Big)^{2\nu_\alpha+1} \frac{\mathcal{M}^2(1+|\lambda_n|)^{4m}}{|\operatorname{Im}\lambda_n|^{2m}} \\
&\leq \Big((\operatorname{Im}(\lambda_n-\alpha))^2 \left(1+\tfrac{1}{K}\right)\Big)^{2\nu_\alpha+1} \frac{\mathcal{M}^2(1+|\lambda_n|)^{4m}}{|\operatorname{Im}\lambda_n|^{2m}} \\
&= \Big((\operatorname{Im}\lambda_n)^2 \left(1+\tfrac{1}{K}\right)\Big)^{2\nu_\alpha+1} \frac{\mathcal{M}^2(1+|\lambda_n|)^{4m}}{|\operatorname{Im}\lambda_n|^{2m}},
\end{aligned}$$

which converges to 0 for $n \to \infty$.

If $\nu_\alpha > 0$, then α is said to be a *generalized pole of non-positive type of τ with multiplicity ν_α*. Assume that there exists a number k_∞ such that (k_∞, ∞) and $(-\infty, -k_\infty)$ are of positive type with respect to τ and let $\nu_\infty \geq 0$ be the smallest integer such that

$$0 \leq \lim_{\lambda \widehat{\to} \infty} \frac{\tau(\lambda)}{\lambda^{2\nu_\infty+1}} < \infty.$$

Again, such an integer ν_∞ always exists. If $\nu_\infty > 0$, then ∞ is said to be a *generalized pole of non-positive type of τ with multiplicity ν_∞*.

Let $\beta \in \mathbb{R}$ and assume, that there exists an open interval I_β, $\beta \in I_\beta$, such that $I_\beta \backslash \{\beta\}$ is of positive type with respect to τ. Suppose that $\lim_{\lambda \widehat{\to} \beta} \frac{\tau(\lambda)}{(\lambda-\beta)^{2\gamma_\beta - 1}}$ exists for some integer $\gamma_\beta \geq 0$ and let $\eta_\beta \geq 0$ be the largest integer such that

$$-\infty < \lim_{\lambda \widehat{\to} \beta} \frac{\tau(\lambda)}{(\lambda - \beta)^{2\eta_\beta - 1}} \leq 0. \tag{5.3}$$

If $\eta_\beta > 0$, then $\beta \in \mathbb{R}$ is said to be a *generalized zero of non-positive type of τ with multiplicity η_β*. Assume that there exists a number $l_\infty > 0$ such that (l_∞, ∞) and $(-\infty, -l_\infty)$ are of positive type with respect to τ, that $\lim_{\lambda \widehat{\to} \infty} \lambda^{2\gamma_\infty - 1}\tau(\lambda)$ exists for some integer $\gamma_\infty \geq 0$ and let $\eta_\infty \geq 0$ be the largest integer such that

$$0 \leq \lim_{\lambda \widehat{\to} \infty} \lambda^{2\eta_\infty - 1}\tau(\lambda) < \infty. \tag{5.4}$$

If $\eta_\infty > 0$, then ∞ is said to be a *generalized zero of non-positive type of τ with multiplicity η_∞*.

Example 5.2. Consider the functions

$$\begin{aligned} &\tau_n : \mathbb{C}\backslash\{0\} \to \mathbb{C}, \quad &&\tau_n(\lambda) = \frac{1}{\lambda^n}, \quad \text{and} \\ &s_n : \mathbb{C} \to \mathbb{C}, &&s_n(\lambda) = \lambda^n. \end{aligned}$$

Using the definitions above, we see that

$$\tau_1, \pm\tau_2, -\tau_3$$

have a generalized pole of non-positive type with multiplicity one at 0 and a generalized zero of non-positive type with multiplicity one at ∞. The polynomials

$$-s_1, \pm s_2, s_3$$

have a generalized zero of non-positive type with multiplicity one at 0 and a generalized pole of non-positive type with multiplicity one at ∞.

The class $\mathcal{N}_0$ of *Nevanlinna functions* is the set of functions $N \in M(\mathbb{C}\backslash\mathbb{R})$ which are holomorphic both in $\mathbb{C}^+$ and $\mathbb{C}^-$ and have non-negative imaginary part on $\mathbb{C}^+$. Note that $N \in \mathcal{N}_0$ has no generalized poles of non-positive type in $\overline{\mathbb{R}}$ and, if N is not identically zero, N has no zeros in $\mathbb{C}^+$ and no generalized zeros of non-positive type in $\overline{\mathbb{R}}$, see [69, Theorem 3.5].

Along with Nevanlinna functions we will deal with generalized Nevanlinna functions $\mathcal{N}_1$

with one negative square. We say that $N_1 \in \mathcal{N}_1$ if there exists $N_0 \in \mathcal{N}_0$ and $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$, such that

$$\begin{aligned} N_1(\lambda) &= \frac{(\lambda-\beta)(\lambda-\overline{\beta})}{(\lambda-\alpha)(\lambda-\overline{\alpha})} N_0(\lambda), \quad \lambda \in \mathbb{C}\backslash\mathbb{R},\, \lambda \notin \{\alpha, \overline{\alpha}\}, \quad \text{or} \\ N_1(\lambda) &= \frac{1}{(\lambda-\alpha)(\lambda-\overline{\alpha})} N_0(\lambda), \quad \lambda \in \mathbb{C}\backslash\mathbb{R},\, \lambda \notin \{\alpha, \overline{\alpha}\}, \quad \text{or} \\ N_1(\lambda) &= (\lambda-\beta)(\lambda-\overline{\beta}) N_0(\lambda), \qquad\qquad \lambda \in \mathbb{C}\backslash\mathbb{R}, \end{aligned}$$

holds. In this case, α is a non-real pole or a generalized pole of non-positive type with multiplicity one of N_1 and β is a non-real zero or a generalized zero of non-positive type with multiplicity one of N_1. Note that in the second equation ∞ is a generalized zero of non-positive type with multiplicity one of N_1 and in the third equation ∞ is a generalized pole of non-positive type with multiplicity one of N_1. Hence, a function $N_1 \in \mathcal{N}_1$, $N_1 \neq 0$, has exactly one zero in $\mathbb{C}^+$ or one generalized zero of non-positive type with multiplicity one in $\overline{\mathbb{R}}$ and N_1 has exactly one pole in $\mathbb{C}^+$ or one generalized pole of non-positive type with multiplicity one in $\overline{\mathbb{R}}$, cf. also [42] and [69, Theorem 3.5].

Example 5.3. We classify the functions of Example 5.2. Since $\pm s_0, s_1$ and $-\tau_1$ have non-negative imaginary part on $\mathbb{C}^+$, they belong to $\mathcal{N}_0$ and we see

$$\tau_1(\lambda) = \frac{1}{\lambda^2} \cdot \lambda, \quad \pm\tau_2(\lambda) = \frac{1}{\lambda^2} \cdot (\pm 1), \quad -\tau_3(\lambda) = \frac{1}{\lambda^2} \cdot \left(-\frac{1}{\lambda}\right), \quad \lambda \in \mathbb{C}\backslash\{0\}.$$

Consequently, $\tau_1, \pm\tau_2, -\tau_3 \in \mathcal{N}_1$ with a generalized pole of non-positive type with multiplicity one at 0 and a generalized zero of non-positive type with multiplicity one at ∞. Likewise,

$$-s_1(\lambda) = \lambda^2 \cdot \left(-\frac{1}{\lambda}\right), \quad \pm s_2(\lambda) = \lambda^2 \cdot (\pm 1), \quad s_3(\lambda) = \lambda^2 \cdot \lambda, \quad \lambda \in \mathbb{C}\backslash\{0\}.$$

Hence, $s_1, \pm s_2, s_3 \in \mathcal{N}_1$ with a generalized zero of non-positive type with multiplicity one at 0 and a generalized pole of non-positive type with multiplicity one at ∞.

The function classes $\mathcal{D}_0$ and $\mathcal{D}_1$ are closely connected to $\mathcal{N}_0$ and $\mathcal{N}_1$, see [24, Section 3.1] (cf. also [24, Theorem 2]).

Definition 5.4. A function $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ belongs to the class $\mathcal{D}_0$ if there exists a point $\lambda_0 \in \mathfrak{h}(\tau)\backslash\{\infty\}$, a Nevanlinna function $N \in \mathcal{N}_0$ holomorphic at λ_0 and a rational function g holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda_0}\}$ such that

$$\frac{\lambda}{(\lambda-\lambda_0)(\lambda-\overline{\lambda_0})}\tau(\lambda) = N(\lambda) + g(\lambda) \tag{5.5}$$

holds for all points λ where τ, N and g are holomorphic. The function $\tau \in M(\mathbb{C}\backslash\mathbb{R})$ belongs to $\mathcal{D}_1$ if (5.5) holds for some $N \in \mathcal{N}_1$.

Since the rational function g is holomorphic in $\overline{\mathbb{C}}\backslash\{\lambda_0, \overline{\lambda_0}\}$ its denominator and enumerator are polynomials of degree at most 2. Moreover, the denominator has at most zeros in λ_0 and $\overline{\lambda}_0$.

Example 5.5. Let τ_n and s_n be the functions of Example 5.2. With Example 5.3 and since $0 \in \mathcal{N}_0$ we see

$$\begin{aligned} \pm s_0,\ \pm s_1,\ s_2,\ \pm\tau_0,\ \pm\tau_1,\ -\tau_2\ &\in \mathcal{D}_0 \quad \text{and} \\ -s_2,\ \pm s_3,\ \tau_2,\ \pm\tau_3\ &\in \mathcal{D}_1. \end{aligned}$$

The next theorem shows that the reciprocal of a $\mathcal{D}_0$-function can be easily classified, see [25, Theorem 2.4].

Theorem 5.6. *Let $\tau \in \mathcal{D}_0$ be not identically zero. Then*

$$-\frac{1}{\tau} \in \mathcal{D}_1$$

if and only if ∞ is not a generalized zero of non-positive type of the function $\lambda \mapsto \lambda^{-1}\tau(\lambda)$ and 0 is not a generalized zero of non-positive type of the function $\lambda \mapsto \lambda\tau(\lambda)$, and

$$-\frac{1}{\tau} \in \mathcal{D}_0$$

otherwise.

Remark 5.7. In this thesis we only need the spaces $\mathcal{N}_0$, $\mathcal{N}_1$, $\mathcal{D}_0$, and $\mathcal{D}_1$. These definitions extend to the classes $\mathcal{N}_\kappa$ and $\mathcal{D}_\kappa$, $\kappa \in \mathbb{N} \cup \{0\}$, see for example [42, 68, 69] and [24, 25].

The following theorem shows the location and sign-types of the poles of a function in $\mathcal{D}_0$ or $\mathcal{D}_1$, see [24, Theorem 2].

Theorem 5.8. *Let $\tau \in M(\mathbb{C}\backslash\mathbb{R})$.*

(i) *$\tau \in \mathcal{D}_0$ if and only if the growth of τ near $\overline{\mathbb{R}}$ is of finite order, there exists a finite set $e \subseteq \mathbb{R}$, such that $(-\infty, 0)\backslash e$ is of negative and $(0,\infty)\backslash e$ is of positive type with respect to τ and*

$$\begin{aligned} \tau &\text{ has no pole in } \mathbb{C}^+, \\ \tau &\text{ has no generalized pole of non-positive type in } (0,\infty), \\ -\tau &\text{ has no generalized pole of non-positive type in } (-\infty,0), \\ 0 &\text{ is no generalized pole of non-positive type of the function } \lambda \mapsto \lambda\tau(\lambda), \text{ and} \\ \infty &\text{ is no generalized pole of non-positive type of the function } \lambda \mapsto \tfrac{1}{\lambda}\tau(\lambda). \end{aligned}$$

(ii) *$\tau \in \mathcal{D}_1$ if and only if the growth of τ near $\overline{\mathbb{R}}$ is of finite order, there exists a finite set $e \subseteq \mathbb{R}$, such that $(-\infty, 0)\backslash e$ is of negative and $(0,\infty)\backslash e$ is of positive type with respect* *to τ and*

> ***either*** *τ has exactly one pole of order one in $\mathbb{C}^+$,*
> ***or*** *τ has exactly one generalized pole of non-positive type with multiplicity one in $(0,\infty)$,*
> ***or*** *$-\tau$ has exactly one generalized pole of non-positive type with multiplicity one in $(-\infty,0)$,*
> ***or*** *0 is a generalized pole of non-positive type with multiplicity one of the function $\lambda \mapsto \lambda\tau(\lambda)$,*
> ***or*** *∞ is a generalized pole of non-positive type with multiplicity one of the function $\lambda \mapsto \frac{1}{\lambda}\tau(\lambda)$.*

The motivation to study $\mathcal{D}_0$- and $\mathcal{D}_1$-functions is given by the following lemma which gives the connection to symmetric relations in Krein spaces, see [24, Lemma 7].

Theorem 5.9. *Let S be a closed symmetric relation in the Krein space $(\mathcal{K},[\cdot,\cdot])$ and assume there exists a selfadjoint extension A_0 with $\kappa = 0$ or $\kappa = 1$ negative squares and $\rho(A_0) \neq \emptyset$ such that $\dim(A_0/S) = 1$. Let $\{\mathbb{C},\Gamma_0,\Gamma_1\}$ be a boundary triplet for S^+ such that $\ker\Gamma_0 = A_0$. Then the corresponding Weyl function M belongs to $\mathcal{D}_{\kappa'}$ with $\kappa' \leq \kappa$.*

Naturally, there is also a connection to rank one perturbations of non-negative operators in Krein spaces: Let A and B be selfadjoint operators in the Krein space $(\mathcal{K},[\cdot,\cdot])$ such that (2.5) holds and assume in addition that A is non-negative. Then B is non-negative or has one negative square, i.e. $\kappa_B = 0$ or $\kappa_B = 1$, respectively, see Theorem 1.13(iii). In this situation the corresponding Weyl functions M_A and M_B (cf. Proposition 2.10) are $\mathcal{D}_0$- or $\mathcal{D}_1$-functions, as the following proposition shows, cf. [17].

Proposition 5.10. *Let A and B be selfadjoint operators in the Krein space $(\mathcal{K},[\cdot,\cdot])$ such that (2.5) holds for some $\lambda_0 \in \rho(A)\cap\rho(B)$. Assume that A is non-negative and let M_A and M_B be given as in Proposition 2.10. Then*

$$M_A \in \mathcal{D}_0 \qquad \text{and} \qquad M_B \in \mathcal{D}_0 \cup \mathcal{D}_1. \tag{5.6}$$

Furthermore the following holds.

(i) *If $M_B \in \mathcal{D}_0$ then all positive (negative) zeros μ of M_A satisfy $M_A'(\mu) > 0$ ($M_A'(\mu) < 0$, respectively).*

(ii) *If $M_B \in \mathcal{D}_1$ then with the possible exception of at most one point μ_0 all positive zeros μ of M_A satisfy $M_A'(\mu) > 0$ and all negative zeros μ of M_A satisfy $M_A'(\mu) < 0$. If this exceptional zero μ_0 is in $\mathbb{R}\backslash\{0\}$, then it is a zero of M_A of at most order three. If it is a zero of order three then $M_A'''(\mu_0) > 0$ for $\mu_0 \in (0,\infty)$ and $M_A'''(\mu_0) < 0$ for $\mu_0 \in (-\infty,0)$.*

(iii) *If there is a positive (negative) zero μ of M_A such that $M_A'(\mu) \leq 0$ ($M_A'(\mu) \geq 0$, respectively), then $M_B \in \mathcal{D}_1$.*

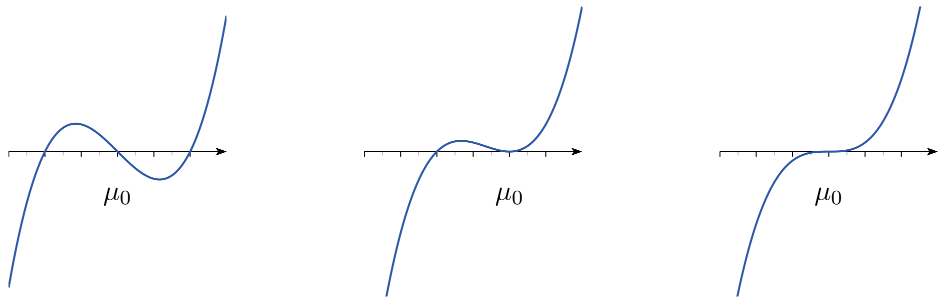

Figure 5.1: Proposition 5.10(ii): Exceptional zero $\mu_0 \in (0, \infty)$ of order one, two, and three.

Proof. The symmetric closed relation $S = A \cap B$ possesses the non-negative extension A and we have $\dim(A/S) = 1$, cf. proof of Proposition 2.9. By Theorem 5.9 $M_A \in D_0$ and with Theorem 5.6, the assertions in (5.6) follow, since M_A is not identically zero by Proposition 2.10(iii). As

$$M_B = -\frac{1}{M_A} \quad \text{on} \quad \rho(A) \cap \rho(B)$$

by Proposition 2.10(iii) and $\rho(A) \cap \rho(B)$ is dense in $\mathbb{C}$, the zeros of M_A correspond to the poles of M_B and vice versa. The order of a zero of M_A equals the order of the corresponding pole of M_B. Moreover, if M_B has a pole of first order at μ then the residue at μ of M_B is given by

$$\operatorname{Res}_\mu(M_B) = \lim_{\lambda \to \mu} (\lambda - \mu) M_B(\lambda) = \lim_{\lambda \to \mu} -\frac{(\lambda - \mu)}{M_A(\lambda)} = \frac{-1}{M_A'(\mu)}.$$

Hence, μ is a zero of first order of M_A.

Let $M_B \in \mathcal{D}_0$. By Theorem 5.8 M_B $(-M_B)$ has no generalized poles of non-positive type in $(0, \infty)$ $((-\infty, 0)$, respectively). Hence, all poles of M_B in $(0, \infty)$ $((-\infty, 0))$ are of first order with negative (positive, respectively) residue (cf. (5.2)) and (i) is shown. Assertion (ii) follows in the same way when taking into account that a function $M_B \in \mathcal{D}_1$ may have at most one pole which is not of first order with negative (positive) residue in $(0, \infty)$ $((-\infty, 0)$, respectively), see Theorem 5.8. Moreover, it also follows from Theorem 5.8 and (5.2) that this exceptional pole μ_0 is at most of order three and that the limit

$$\lim_{\lambda \to \mu_0} (\lambda - \mu_0)^3 M_B(\lambda)$$

exists and is non-positive (non-negative) if μ_0 is in $(0, \infty)$ $((-\infty, 0)$, respectively). This shows (ii). Finally, we see by the above reasoning that if μ is a positive (negative) zero of M_A with $M_A'(\mu) \leq 0$ $(M_A'(\mu) \geq 0$, respectively) then M_B has a pole at μ which is not of first order with a negative (positive, respectively) residue in $(0, \infty)$ $((-\infty, 0)$, respectively). As $M_B \in \mathcal{D}_0 \cup \mathcal{D}_1$ by (5.6) we conclude $M_B \in \mathcal{D}_1$ from Theorem 5.8(ii). $\square$

The next lemma provides some more properties of the function M_A at the point 0.

Lemma 5.11. *Let the assumptions be as in Proposition 5.10. Then the following holds.*

(i) *If* 0 *is a pole of* M_A *then* 0 *is a pole of first or second order. If* 0 *is a pole of second order then*

$$\lim_{\lambda \nearrow 0} M_A(\lambda) = \lim_{\lambda \searrow 0} M_A(\lambda) = -\infty.$$

(ii) *If* $M_B \in \mathcal{D}_1$ *and* M_A *is holomorphic at* 0 *then*

$$M_A(0) > 0.$$

(iii) *Assume* M_A *is holomorphic at* 0 *and let* 0 *be a zero of* M_A*. Then* 0 *is a zero of at most second order and in this case we have*

$$M_A''(0) > 0.$$

Proof. (i) Let 0 be a pole of M_A. As $M_A \in \mathcal{D}_0$, the function $\lambda \mapsto \lambda M_A(\lambda)$ either is holomorphic at 0 or has a pole of first order at 0, see Theorem 5.8(i). In the second case we have

$$-\infty < \lim_{\lambda \widehat{\to} 0} \lambda^2 M_A(\lambda) = \lim_{\lambda \to 0} \lambda^2 M_A(\lambda) < 0,$$

cf. (5.2), as $\lim \lambda^2 M_A(\lambda) = 0$ implies that $\lambda \mapsto \lambda M_A(\lambda)$ is holomorphic at 0. Hence, 0 is a pole of at most second order of M_A and if 0 is a pole of second order of M_A it satisfies

$$-\infty < \lim_{\lambda \to 0} \lambda^2 M_A(\lambda) < 0,$$

which yields (i).

(ii) If $M_B \in \mathcal{D}_1$ then Theorem 5.6 implies that 0 is not a generalized zero of non-positive type of $\lambda \mapsto \lambda M_A(\lambda)$. Under the assumption that M_A is holomorphic at 0, this is equivalent to (ii), cf. (5.3).

(iii) Consider (5.5) with $\lambda_0 = 0$:

$$\lambda^{-1} M_A(\lambda) = N_A(\lambda) + g_A(\lambda), \tag{5.7}$$

where N_A is a Nevanlinna function holomorphic at 0 and g_A is a rational function holomorphic in $\mathbb{C}\backslash\{0\}$ with a possible pole at 0. Assume

$$M_A(0) = M_A'(0) = 0. \tag{5.8}$$

Then the left hand side of (5.7) is holomorphic at 0 and hence g_A is bounded on $\mathbb{C}$, which implies that g_A is equal to a (real) constant c. Then (5.7) becomes

$$M_A(\lambda) = \lambda \left(N_A(\lambda) + c \right). \tag{5.9}$$

Hence, we have $M_A'(\lambda) = N_A(\lambda) + c + \lambda N_A'(\lambda)$ and $M_A''(\lambda) = 2N_A'(\lambda) + \lambda N_A''(\lambda)$. In particular

$$M_A'(0) = N_A(0) + c \quad \text{and} \quad M_A''(0) = 2N_A'(0).$$

It follows from (5.8) that the function $N_A + c$ vanishes at 0. Non-constant Nevanlinna functions have a positive derivative in real points of holomorphy, which follows from the integral representation, cf. [1, Theorem 59.2]. Here, $N_A + c$ is not identically zero, as this would by (5.9) imply $M_A \equiv 0$, which is a contradiction to Proposition 2.10(iii). We conclude

$$M_A''(0) = 2(N_A + c)'(0) > 0,$$

and hence 0 is a zero of at most second order of M_A. □

5.2 Total Number of Eigenvalues in an Interval

For an interval $I \subseteq \mathbb{R}$ we denote the numbers of distinct eigenvalues of A and B in I by $n_A(I)$ and $n_B(I)$, respectively,

$$n_A(I) = \#\big\{\lambda \,|\, \lambda \in I \cap \sigma_{\mathrm{p}}(A)\big\} \quad \text{and} \quad n_B(I) = \#\big\{\lambda \,|\, \lambda \in I \cap \sigma_{\mathrm{p}}(B)\big\},$$

where $\#\Omega$ stands for the cardinality of a set Ω. We set for the common eigenvalues of A and B in I

$$n_{A,B}(I) = \#\big\{\lambda \,|\, \lambda \in I \cap \sigma_{\mathrm{p}}(A) \cap \sigma_{\mathrm{p}}(B)\big\}.$$

Here, multiplicities of eigenvalues are not counted. Theorem 5.12 below provides sharp estimates from below and above on the number of distinct eigenvalues of B in terms of the number of distinct eigenvalues of A. The last assertion on the infinite number of distinct eigenvalues of A and B in I can be viewed as a special case of [21, Theorem 4.3].

Theorem 5.12. *Let A, B, and I be as in Assumption* I *and assume in addition that A is non-negative. Then B is non-negative ($\kappa_B = 0$) or has one negative square ($\kappa_B = 1$) and if $n_A(I) < \infty$ the following estimates hold.*

(i) *If $0 \notin I$ then*

$$n_A(I) - n_{A,B}(I) - 1 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(ii) *If $0 \in I$ then*

$$n_A(I) - n_{A,B}(I) - 2 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + \begin{cases} 2 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

Each of the estimates in (i) *and* (ii) *is sharp. Moreover, $n_A(I) = \infty$ if and only if $n_B(I) = \infty$.*

The upper and lower estimates in the next corollary follow from $n_{A,B}(I) \leq n_A(I)$ and $-n_B(I) \leq -n_{A,B}(I)$, respectively.

Corollary 5.13. *Let the assumptions be as in Theorem 5.12. Then the following estimates hold.*

(i) *If* $0 \notin I$ *then*

$$\frac{n_A(I)-1}{2} \leq n_B(I) \leq 2n_A(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(ii) *If* $0 \in I$ *then*

$$\frac{n_A(I)-2}{2} \leq n_B(I) \leq 2n_A(I) + \begin{cases} 2 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

Each of the estimates in (i) *and* (ii) *is sharp.*

The next corollary treats the case $n_{A,B}(I) = 0$ and will play an important role in the proof of Theorem 5.18 in Section 5.3.

Corollary 5.14. *Let the assumptions be as in Theorem 5.12 and assume in addition that* $I \cap \sigma_{\mathrm{p}}(A) \cap \sigma_{\mathrm{p}}(B) = \emptyset$*. Then the following estimates hold.*

(i) *If* $0 \notin I$ *then*

$$n_A(I) - 1 \leq n_B(I) \leq n_A(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(ii) *If* $0 \in I$ *then*

$$n_A(I) - 2 \leq n_B(I) \leq n_A(I) + \begin{cases} 2 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

Each of the estimates in (i) *and* (ii) *is sharp.*

Proof of Theorem 5.12. The proof is divided into eight separate steps. In Steps 1 and 2 the lower estimates are shown and in Steps 3 - 5 the upper estimates are verified. The sharpness of the estimates is shown in Steps 6 and 7 for two particularly interesting situations; from the construction it is clear how the sharpness of the remaining estimates follows. Finally, in Step 8 we verify the assertion on the infiniteness of the eigenvalues.

Step 1. Lower estimate in (i). We verify the estimate

$$n_A(I) - n_{A,B}(I) - 1 \leq n_B(I). \tag{5.10}$$

By assumption $0 \notin I$ and we have $I \subseteq (0, \infty)$ or $I \subseteq (-\infty, 0)$. We discuss the case $I \subseteq (0, \infty)$ only, the case $I \subseteq (-\infty, 0)$ follows analogously. Then, as A is non-negative, all eigenvalues of A in I are of positive type, that is $\sigma(A) \cap I \subseteq \sigma_{++}(A)$, cf. Corollary 1.14(i). As $n_A(I) < \infty$ we have $n_{A,B}(I) < \infty$. If $n_A(I) - 1 - n_{A,B}(I) \leq n_{A,B}(I)$ then the estimate

(5.10) holds since $n_{A,B}(I) \leq n_B(I)$. If $n_A(I) - 1 - n_{A,B}(I) > n_{A,B}(I)$ then there exist at least $n_A(I)-1-2n_{A,B}(I)$ pairs of eigenvalues in $\sigma_{++}(A)\cap\rho(B)$ to which Proposition 2.15(i) can be applied. This leads to $n_A(I) - 1 - 2n_{A,B}(I)$ eigenvalues of B in $\rho(A) \cap I$ and since there are also $n_{A,B}(I)$ eigenvalues of B in $\sigma(A) \cap I$ we obtain the estimate (5.10).

Step 2. Lower estimate in (ii). Let $0 \in I$ and set $I_\pm = I \cap \mathbb{R}^\pm$. In order to show the estimate

$$n_A(I) - n_{A,B}(I) - 2 \leq n_B(I) \tag{5.11}$$

observe that by Step 1 the estimates

$$n_A(I_\pm) - n_{A,B}(I_\pm) - 1 \leq n_B(I_\pm) \tag{5.12}$$

hold. Clearly,

$$n_A(I_+) + n_A(I_-) = \begin{cases} n_A(I) & \text{if } 0 \notin \sigma_\mathrm{p}(A), \\ n_A(I) - 1 & \text{if } 0 \in \sigma_\mathrm{p}(A) \end{cases}$$

and

$$n_{A,B}(I_+) + n_{A,B}(I_-) = \begin{cases} n_{A,B}(I) & \text{if } 0 \notin \sigma_\mathrm{p}(A) \cap \sigma_\mathrm{p}(B), \\ n_{A,B}(I) - 1 & \text{if } 0 \in \sigma_\mathrm{p}(A) \cap \sigma_\mathrm{p}(B). \end{cases}$$

Together with (5.12) this yields

$$\begin{aligned} n_B(I) &= \begin{cases} n_B(I_+) + n_B(I_-) & \text{if } 0 \notin \sigma_\mathrm{p}(B), \\ n_B(I_+) + n_B(I_-) + 1 & \text{if } 0 \in \sigma_\mathrm{p}(B), \end{cases} \\ &\geq \begin{cases} n_A(I_+) - n_{A,B}(I_+) + n_A(I_-) - n_{A,B}(I_-) - 2 & \text{if } 0 \notin \sigma_\mathrm{p}(B), \\ n_A(I_+) - n_{A,B}(I_+) + n_A(I_-) - n_{A,B}(I_-) - 1 & \text{if } 0 \in \sigma_\mathrm{p}(B), \end{cases} \\ &= \begin{cases} n_A(I) - n_{A,B}(I) - 2 & \text{if } 0 \notin \sigma_\mathrm{p}(B),\ 0 \notin \sigma_\mathrm{p}(A), \\ n_A(I) - n_{A,B}(I) - 3 & \text{if } 0 \notin \sigma_\mathrm{p}(B),\ 0 \in \sigma_\mathrm{p}(A), \\ n_A(I) - n_{A,B}(I) - 1 & \text{if } 0 \in \sigma_\mathrm{p}(B),\ 0 \notin \sigma_\mathrm{p}(A), \\ n_A(I) - n_{A,B}(I) - 1 & \text{if } 0 \in \sigma_\mathrm{p}(B),\ 0 \in \sigma_\mathrm{p}(A). \end{cases} \end{aligned}$$

It remains to show estimate (5.11) in the case $0 \in \sigma_\mathrm{p}(A)$ and $0 \notin \sigma_\mathrm{p}(B)$. Assume first that $I_- \cap \sigma(A)$ is empty. Then $n_B(I_-) \geq 0$, $n_A(I_+) = n_A(I) - 1$, and (5.12) yield

$$n_B(I) \geq n_B(I_+) \geq n_A(I_+) - n_{A,B}(I_+) - 1 = n_A(I) - n_{A,B}(I) - 2,$$

that is, (5.11) holds. A similar reasoning implies (5.11) for the case that $I_+ \cap \sigma(A)$ is empty. Now we assume $I_\pm \cap \sigma(A) \neq \emptyset$. Denote by λ_- the largest eigenvalue of A in I_- and by λ_+ the smallest eigenvalue of A in I_+. Assume first $\lambda_- \in \sigma_\mathrm{p}(B)$ and apply the lower estimate from Step 1 to the intervals $I_{\lambda_-} := (-\infty, \lambda_-) \cap I_-$ and I_+:

$$\begin{aligned} n_B(I) &= n_B(I_{\lambda_-}) + n_B([\lambda_-, 0]) + n_B(I_+) \\ &\geq n_A(I_{\lambda_-}) - n_{A,B}(I_{\lambda_-}) - 1 + n_B([\lambda_-, 0]) + n_A(I_+) - n_{A,B}(I_+) - 1 \\ &= n_A(I_{\lambda_-}) + n_A(I_+) - \big(n_{A,B}(I_{\lambda_-}) + n_{A,B}(I_+)\big) + n_B([\lambda_-, 0]) - 2. \end{aligned}$$

In the present situation we have

$$\begin{aligned} n_A(I) &= n_A(I_{\lambda_-}) + n_A([\lambda_-, 0]) + n_A(I_+) = n_A(I_{\lambda_-}) + 2 + n_A(I_+), \\ n_{A,B}(I) &= n_{A,B}(I_{\lambda_-}) + n_{A,B}([\lambda_-, 0]) + n_{A,B}(I_+) = n_{A,B}(I_{\lambda_-}) + 1 + n_{A,B}(I_+) \end{aligned}$$

and hence we obtain

$$\begin{aligned} n_B(I) &\geq n_A(I) - 2 - \big(n_{A,B}(I) - 1\big) + n_B([\lambda_-, 0]) - 2 \\ &= n_A(I) - n_{A,B}(I) + n_B([\lambda_-, 0]) - 3. \end{aligned}$$

Together with $n_B([\lambda_-, 0]) \geq 1$ we conclude (5.11). In a similar way the estimate (5.11) follows if $\lambda_+ \in \sigma_{\mathrm{p}}(B)$. Thus it remains to show (5.11) for $0 \in \sigma_{\mathrm{p}}(A)$, $0 \notin \sigma_{\mathrm{p}}(B)$, and $\lambda_\pm \notin \sigma_{\mathrm{p}}(B)$. For this we consider the function $M_A : \rho(A) \to \mathbb{C}$ from Proposition 2.10 which is continuous and real-valued on $\rho(A) \cap \mathbb{R}$. By Corollary 2.11(ii) the point 0 is a pole of M_A and by Lemma 5.11(i) it is of first or of second order. If 0 is a pole of first order we conclude from $\lambda_- \in \sigma_{--}(A)$, $\lambda_+ \in \sigma_{++}(A)$, and Lemma 2.14 that M_A has a zero either in $(\lambda_-, 0)$ or in $(0, \lambda_+)$, and hence an eigenvalue of B, cf. Corollary 2.11(i). If 0 is a pole of second order, then M_A has zeros (and, hence, eigenvalues of B) in both intervals $(\lambda_-, 0)$ and $(0, \lambda_+)$, cf. Lemma 5.11(i), Corollary 2.11(i), and Lemma 2.14. Thus in both cases there is at least one eigenvalue of B in the interval (λ_-, λ_+). Therefore, for $\epsilon > 0$ sufficiently small we conclude

$$n_B([\lambda_- + \epsilon, \lambda_+ - \epsilon]) \geq 1, \qquad \lambda_- + \epsilon < 0 < \lambda_+ - \epsilon. \tag{5.13}$$

Let us apply the lower estimate from Step 1 to $I_{\lambda_-+\epsilon} = (-\infty, \lambda_- + \epsilon) \cap I_-$ and $I_{\lambda_+-\epsilon} = (\lambda_+ - \epsilon, \infty) \cap I_+$. Then, with (5.13) we obtain

$$\begin{aligned} n_B(I) &= n_B(I_{\lambda_-+\epsilon}) + n_B([\lambda_- + \epsilon, \lambda_+ - \epsilon]) + n_B(I_{\lambda_+-\epsilon}) \\ &\geq n_A(I_{\lambda_-+\epsilon}) - n_{A,B}(I_{\lambda_-+\epsilon}) - 1 + n_B([\lambda_- + \epsilon, \lambda_+ - \epsilon] \\ &\quad + n_A(I_{\lambda_+-\epsilon}) - n_{A,B}(I_{\lambda_+-\epsilon}) - 1 \\ &\geq n_A(I_{\lambda_-+\epsilon}) - n_{A,B}(I_{\lambda_-+\epsilon}) + n_A(I_{\lambda_+-\epsilon}) - n_{A,B}(I_{\lambda_+-\epsilon}) - 1 \\ &= n_A(I) - n_A([\lambda_- + \epsilon, \lambda_+ - \epsilon]) - (n_{A,B}(I) - n_{A,B}([\lambda_- + \epsilon, \lambda_+ - \epsilon])) - 1. \end{aligned}$$

In the present setting we have $n_A([\lambda_- + \epsilon, \lambda_+ - \epsilon]) = 1$ and $n_{A,B}([\lambda_- + \epsilon, \lambda_+ - \epsilon]) = 0$. This implies the estimate (5.11).

Step 3. Upper estimate in (i) and (ii) if $\kappa_{\mathbf{B}} = \mathbf{0}$. If B is non-negative these two estimates follow immediately from (5.10) and (5.11) by interchanging the roles of A and B.

Step 4. Upper estimate in (i) if $\kappa_{\mathbf{B}} = \mathbf{1}$. We show that the inequality

$$n_B(I) \leq n_A(I) + n_{A,B}(I) + 3 \tag{5.14}$$

holds if $0 \notin I$ and B has one negative square. Let us again discuss the case $I \subseteq (0,\infty)$ only, the case $I \subseteq (-\infty,0)$ follows analogously. Since $I \cap \sigma(A)$ consists of $n_A(I)$ distinct eigenvalues the set $I \cap \rho(A)$ consists of $n_A(I)+1$ open subintervals I_k, $1 \leq k \leq n_A(I)+1$. We use that M_A is continuous and real-valued on each subinterval I_k, and that by Corollary 2.11(i) the zeros of M_A in I_k coincide with the eigenvalues of B in I_k. As $\kappa_B = 1$ there is at most one point $\nu \in \sigma_{\mathrm{p}}(B) \cap I$ with $\nu \notin \sigma_{++}(B)$ by Corollary 1.14(ii). If $\nu \in \sigma_{\mathrm{p}}(A)$ then $I_k \cap \sigma(B)$, $1 \leq k \leq n_A(I)+1$, is contained in $\sigma_{++}(B)$ according to Corollary 1.14(ii) and each zero μ in I_k of M_A satisfies $M_A'(\mu) > 0$ by Lemma 2.13(i). Thus in each subinterval I_k, $1 \leq k \leq n_A(I)+1$, there is at most one eigenvalue of B so that the set $I \cap \rho(A)$ contains at most $n_A(I)+1$ eigenvalues of B. Clearly, the set $I \cap \sigma(A)$ contains $n_{A,B}(I)$ eigenvalues of B and hence $n_B(I) \leq n_A(I) + n_{A,B}(I) + 1$. In particular, (5.14) follows in the case $\nu \in \sigma_{\mathrm{p}}(A)$. It remains to show estimate (5.14) in the case $\nu \in \rho(A)$. Then ν belongs to some subinterval I_j for some j with $1 \leq j \leq n_A(I)+1$ and the function M_A satisfies $M_A'(\nu) \leq 0$ by Lemma 2.13(i). Since all other eigenvalues μ of B in $I \cap \rho(A)$ belong to $\sigma_{++}(B)$ it follows from Lemma 2.13(i) that $M_A'(\mu) > 0$. Hence in I_j there are at most three eigenvalues of B and in each of the subintervals I_k, $1 \leq k \leq n_A(I)+1$, $k \neq j$, there is at most one eigenvalue of B. Summing up it follows that the set $I \cap \rho(A)$ contains at most $n_A(I)+3$ eigenvalues and, as $I \cap \sigma(A)$ contains $n_{A,B}(I)$ eigenvalues of B, (5.14) is shown.

Step 5. Upper estimate in (ii) if $\kappa_{\mathbf{B}} = \mathbf{1}$. In this step we discuss the case $0 \in I$ and B has one negative square. We verify the inequality

$$n_B(I) \leq n_A(I) + n_{A,B}(I) + 3. \tag{5.15}$$

In order to show this we consider again the open subintervals I_k, $1 \leq k \leq n_A(I)+1$, as in Step 4. Assume that $0 \in \sigma_{\mathrm{p}}(A)$. Then the arguments used in the proof of Step 4 remain valid and it follows that in at most one interval I_j there might be at most three zeros of M_A, in all other intervals I_k there is at most one zero. This implies (5.15) if $0 \in \sigma_{\mathrm{p}}(A)$. Let us now discuss the case $0 \in \rho(A)$ so that $0 \in I_j$ for some j. If M_A has two or three zeros in one of the other subintervals I_k, $k \neq j$, then according to Lemma 2.13(i)-(ii) one of these zeros is an eigenvalue μ of B which does not belong to $\sigma_{++}(B)$ ($\sigma_{--}(B)$) if $I_k \subseteq (0,\infty)$ ($I_k \subseteq (-\infty,0)$, respectively). Moreover, by Proposition 5.10(iii) the function M_B belongs to the class $\mathcal{D}_1$ and by Lemma 5.11(ii) we have $M_A(0) > 0$. But this implies that there are no zeros of M_A in I_j as otherwise $M_A'(\mu_-) \geq 0$ for some $\mu_- < 0$ in I_j or $M_A'(\mu_+) \leq 0$ for some $\mu_+ > 0$ in I_j, which is impossible by Proposition 5.10(ii). Hence, if $0 \in I_j$ and M_A has two or three zeros in one of the other subintervals I_k then (5.15) is valid. It remains to discuss the case $0 \in I_j$ and M_A has at most one zero in each of the other subintervals I_k, $k \neq j$. Suppose that $M_A(0) > 0$. By Proposition 5.10(i) and (ii) there are at most two zeros of M_A in I_j and (5.15) is true for $M_A(0) > 0$. In the case $M_A(0) = 0$ three other zeros in I_j would imply $M_B \in \mathcal{D}_1$ by Proposition 5.10(iii) and hence $M_A(0) > 0$ by Lemma 5.11(ii). Thus only two zeros in $I_j \backslash \{0\}$ may exist and (5.15) holds also in the case $M_A(0) = 0$. Finally, if $M_A(0) < 0$ then again three zeros in I_j would imply $M_B \in \mathcal{D}_1$ by Proposition 5.10(iii) and hence $M_A(0) > 0$ by Lemma 5.11(ii). Thus, also in this case there are at most two zeros of M_A in I_j. We have proved (5.15).

Step 6. Sharpness of the upper estimate in (i) if $\kappa_{\mathbf{B}} = \mathbf{1}$. We discuss the case $0 \notin I$.

Our aim is to show that the estimate

$$n_B(I) \leq n_A(I) + n_{A,B}(I) + 3 \tag{5.16}$$

is sharp. For this we show that there exist matrices A, B, and an open interval I such that Assumption I is satisfied and equality holds in (5.16). Here we give an idea how to construct specific examples fitting to a given eigenvalue distribution. For explicit examples, see Examples 5.15 and 5.16. Let $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \lambda_{n+1}$ for some $n \in \mathbb{N}$ and define $I := (\lambda_0, \lambda_{n+1})$. Choose a rational function M symmetric with respect to the real axis such that

- M has poles of first order at 0 and at each λ_j, $j = 0, \ldots, n+1$. These are the only poles of M and M is monotonously increasing in every interval $(\lambda_1, \lambda_2), \ldots, (\lambda_n, \lambda_{n+1})$.
- M has three zeros $\mu_{-1} < \mu_0 < \mu_1$ in the interval (λ_0, λ_1) such that $M'(\mu_{-1}) > 0$, $M'(\mu_0) < 0$, and $M'(\mu_1) > 0$.
- $\lim_{x \to \pm\infty} M(x) \in \mathbb{R} \backslash \{0\}$.
- $M \in \mathcal{D}_0$ and $-M^{-1} \in \mathcal{D}_1$.

Then M has a zero μ_j in every interval $(\lambda_{j-1}, \lambda_j)$, $j = 2, \ldots, n+1$. Such a function is given by

$$M(\lambda) = -\frac{\prod_{j=-1}^{n+1} (\lambda - \mu_j)}{\lambda \prod_{j=0}^{n+1} (\lambda - \lambda_j)} \tag{5.17}$$

and an example for $n = 0$ is the function M_1 in Figure 5.2. We show the last assertion on M with Theorem 5.8, the other claims are evident. From the definition it is clear, that M has no poles in $\mathbb{C}^+$ and $-M$ has no generalized poles of non-positive type in $(-\infty, 0)$. We show that M has no generalized poles of non-positive type in $(0, \infty)$. For this, let $k \in \{1, \ldots, n+1\}$. Then

$$\begin{aligned}
\lim_{\lambda \mathbin{\hat{\to}} \lambda_k} (\lambda - \lambda_k) M(\lambda) &= - \lim_{\lambda \mathbin{\hat{\to}} \lambda_k} \frac{\prod_{j=-1}^{k} (\lambda - \mu_j)}{\lambda \prod_{j=0}^{k-1} (\lambda - \lambda_j)} \frac{\prod_{j=k+1}^{n+1} (\lambda - \mu_j)}{\prod_{j=k+1}^{n+1} (\lambda - \lambda_j)} \\
&= - \frac{\prod_{j=-1}^{k} (\lambda_k - \mu_j)}{\lambda_k \prod_{j=0}^{k-1} (\lambda_k - \lambda_j)} \frac{(-1)^{n+1-k} \prod_{j=k+1}^{n+1} (\mu_j - \lambda_k)}{(-1)^{n+1-k} \prod_{j=k+1}^{n+1} (\lambda_j - \lambda_k)} < 0,
\end{aligned}$$

and hence λ_k, $k = 1, \ldots, n+1$, is no generalized pole of non-positive type of M. An analogous calculation shows that also λ_0 is no generalized pole of non-positive type of M.

Moreover, 0 is no generalized pole of non-positive type of the function $\lambda \mapsto \lambda M(\lambda)$ since

$$\lim_{\lambda \hat{\to} 0} \lambda \cdot \lambda M(\lambda) = - \lim_{\lambda \hat{\to} 0} \lambda \frac{\prod_{j=-1}^{n+1} (\lambda - \mu_j)}{\prod_{j=0}^{n+1} (\lambda - \lambda_j)} = 0. \tag{5.18}$$

Finally, ∞ is no generalized pole of non-positive type of the function $\lambda \mapsto \frac{1}{\lambda} M(\lambda)$ as

$$\lim_{\lambda \hat{\to} \infty} \frac{1}{\lambda} \cdot \frac{M(\lambda)}{\lambda} = - \lim_{\lambda \hat{\to} \infty} \frac{1}{\lambda^3} \frac{\prod_{j=-1}^{n+1} (\lambda - \mu_j)}{\prod_{j=0}^{n+1} (\lambda - \lambda_j)} = 0. \tag{5.19}$$

Thus $M \in \mathcal{D}_0$ by Theorem 5.8(i). By Theorem 5.6 $-M^{-1} \in \mathcal{D}_1$ if and only if 0 is no generalized zero of non-positive type of $\lambda \mapsto \lambda M(\lambda)$ and ∞ is no generalized zero of $\lambda \mapsto \frac{1}{\lambda} M(\lambda)$. We see that

$$\lim_{\lambda \hat{\to} 0} \frac{\lambda M(\lambda)}{\lambda} = - \lim_{\lambda \hat{\to} 0} \frac{\prod_{j=-1}^{n+1} (\lambda - \mu_j)}{\lambda \prod_{j=0}^{n+1} (\lambda - \lambda_j)}$$

does not exist. Together with (5.18) this shows that 0 is no generalized zero of non-positive type of $\lambda \mapsto \lambda M(\lambda)$. Furthermore, $\lim_{\lambda \hat{\to} \infty} M(\lambda) < 0$ and together with (5.19) we see that ∞ is no generalized pole of non-positive type of $\lambda \mapsto \frac{1}{\lambda} M(\lambda)$, cf. (5.4). Thus, $-M^{-1} \in \mathcal{D}_1$.

According to Theorem 2.25 (see also [16, Corollary 3.5]) there exists a finite-dimensional Krein space $(\mathcal{K}, [\cdot, \cdot])$, a (non-densely defined) closed symmetric operator S and a boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for the adjoint S^+ such that the corresponding Weyl function coincides with M and $A := \ker \Gamma_0$ is a matrix, i.e. A has no multivalued part as M has no pole at $\pm\infty$, see also [54, 68]. Moreover, $\sigma(A)$ coincides with the poles of M. By Lemma 2.8 $\{\mathbb{C}, \Gamma_1, -\Gamma_0\}$ is a boundary triplet for S^+ with Weyl function $-M^{-1}$. Let $B := \ker \Gamma_1$. Then B is a selfadjoint matrix since $-M^{-1}$ has no pole at $\pm\infty$ and $\kappa_B = 1$ by Proposition 5.10(iii) and Theorem 5.9. As both A and B are selfadjoint extensions of the symmetric (non-densely defined) matrix S with $\dim(A/S) = \dim(B/S) = 1$, the difference of A and B and of their resolvents is a rank one operator, so that Assumption I is satisfied. Moreover, the zeros of M in I coincide with $\sigma(B) \cap I$. Hence B has 3 eigenvalues in the interval (λ_0, λ_1) and one eigenvalue in each of the n intervals $(\lambda_1, \lambda_2), \ldots, (\lambda_n, \lambda_{n+1})$, that is, $n_B(I) = n + 3$ and equality in (5.16) is shown for the case $n_{A,B}(I) = 0$. In order to obtain a sharp estimate in the remaining cases add orthogonally to A and B a non-negative matrix C such that $\sigma_p(C) \subseteq \sigma_p(A)$. Then,

$$\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \tag{5.20}$$

differ by a rank one matrix and have $n_C(I)$ common eigenvalues in the interval I. This shows that (5.16) is sharp.

Step 7. Sharpness of the lower estimate in (ii). In order to show that for $0 \in I$ the estimate

$$n_A(I) - n_{A,B}(I) - 2 \leq n_B(I) \tag{5.21}$$

is sharp let $\lambda_0 < 0 < \lambda_1 < \cdots < \lambda_n$ with $n \in \mathbb{N}$ and consider a rational function M such that

- M has poles of first order at each λ_i. These are the only poles of M and M is monotonously increasing in every interval $(\lambda_1, \lambda_2), \ldots, (\lambda_{n-1}, \lambda_n)$.
- M is positive in the interval (λ_0, λ_1).
- $\lim_{x \to \pm\infty} M(x) \in \mathbb{R}\backslash\{0\}$ and $M \in \mathcal{D}_0$.

Then M has a zero μ_j in every interval $(\lambda_{j-1}, \lambda_j)$, $j = 2, \ldots, n$. Such a function is given by

$$M(\lambda) = -\frac{(\lambda^2 + 1) \prod_{j=2}^{n} (\lambda - \mu_j)}{\prod_{j=0}^{n} (\lambda - \lambda_j)}.$$

Indeed, for $\lambda \in (\lambda_0, \lambda_1)$ we have

$$M(\lambda) = -\frac{(\lambda^2 + 1)(-1)^{n-1} \prod_{j=2}^{n} (\mu_j - \lambda)}{(\lambda - \lambda_0)(-1)^n \prod_{j=0}^{n} (\lambda_j - \lambda)} > 0.$$

Moreover, M has no poles in $\mathbb{C}^+$ and

$$\lim_{\lambda \,\widehat{\to}\, \lambda_0} -(\lambda - \lambda_0) M(\lambda) = \frac{(\lambda^2 + 1)(-1)^{n-1} \prod_{j=2}^{n} (\mu_j - \lambda_0)}{(-1)^n \prod_{j=1}^{n} (\lambda_j - \lambda)} < 0,$$

which shows that $-M$ has no generalized poles of non-positive type in $(-\infty, 0)$. An analogous calculation shows that M has no generalized poles of non-positive type in $(0, \infty)$. Since $\lim_{\lambda \to 0} \lambda^2 M(\lambda) = 0 = \lim_{\lambda \to \infty} \frac{1}{\lambda^2} M(\lambda)$, we see that 0 is no generalized pole of non-positive type of $\lambda \mapsto \lambda M(\lambda)$ and ∞ is no generalized pole of non-positive type of $\lambda \mapsto \frac{1}{\lambda} M(\lambda)$. By Theorem 5.8 $M \in \mathcal{D}_0$.

As in Step 6 there exists a Krein space and selfadjoint matrices A and B which differ by a rank one matrix such that λ_i, $i = 0, \ldots, n$, are eigenvalues of A and μ_j, $j = 2, \ldots, n$, are eigenvalues of B. Hence for $\epsilon > 0$ sufficiently small A has $n + 1$ distinct eigenvalues in the interval $I = (\lambda_0 - \epsilon, \lambda_n + \epsilon)$ and B has $n - 1$ eigenvalues in I, that is, (5.21) is sharp if $n_{A,B}(I) = 0$. In the case $n_{A,B}(I) > 0$ one obtains that (5.21) is sharp by adding orthogonally a suitable non-negative matrix C as in (5.20).

Step 8. Proof of $\mathbf{n_A(I) = \infty}$ if and only if $\mathbf{n_B(I) = \infty}$. If $n_{A,B}(I) = \infty$ then $n_B(I) = \infty = n_A(I)$ and the assertion is true. If $n_A(I) = \infty$ and $n_{A,B}(I) < \infty$ then there are infinitely many pairs of eigenvalues in $\sigma_{++}(A)$ or $\sigma_{--}(A)$ to which Proposition 2.15(i) or (ii) can be applied. This yields $n_B(I) = \infty$. Conversely, if $n_B(I) = \infty$ then the same reasoning implies $n_A(I) = \infty$ and the assertion is proved. $\square$

Example 5.15. Define the function M_1 by

$$M_1(\lambda) = -\frac{(\lambda-2)(\lambda-3)(\lambda-4)}{\lambda(\lambda-1)(\lambda-5)},$$

cf. Figure 5.2. Then M_1 is of the form (5.17) and hence $M_1 \in \mathcal{D}_0$ and $-M^{-1} \in \mathcal{D}_1$. We have

$$M_1(\lambda) = \frac{24}{5\lambda} - \frac{3}{2(\lambda-1)} - \frac{3}{10(\lambda-5)} - 1.$$

With Propositions 2.17 and 2.22 and 2.23 we construct a realisation of M_1: Set

$$c_1 = \frac{24}{5}, \quad c_2 = \frac{3}{2}, \quad \text{and} \quad c_3 = \frac{3}{10}$$

and equip $\mathbb{C}^3$ with the indefinite inner product

$$[\cdot,\cdot] := \langle J\cdot,\cdot\rangle, \quad J = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}. \tag{5.22}$$

We define the symmetric relation S in $\mathbb{C}^3$ according to Propositions 2.17 and 2.22 and 2.23 by

$$S := \Big\{ \{f,f'\} \in \mathbb{C}^3 \times \mathbb{C}^3 \,\Big|\, \tfrac{-1}{\sqrt{c_1}} f_1' = \tfrac{1}{\sqrt{c_2}}(f_2 - f_2') = \tfrac{1}{\sqrt{c_3}}(5f_3 - f_3') = 0, \\ -\sqrt{c_1} f_1 + \sqrt{c_2} f_2 + \sqrt{c_3} f_3 - \tfrac{1}{\sqrt{c_3}}(5f_3 - f_3') = 0 \Big\}.$$

Then

$$S^+ = \Big\{ \{f,f'\} \in \mathbb{C}^3 \times \mathbb{C}^3 \,\Big|\, \tfrac{-1}{\sqrt{c_1}} f_1' = \tfrac{1}{\sqrt{c_2}}(f_2 - f_2') = \tfrac{1}{\sqrt{c_3}}(5f_3 - f_3') \Big\}$$

and the triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ given by

$$\Gamma_0 : S^+ \to \mathbb{C}, \quad \Gamma_0(\{f,f'\}) = \tfrac{-1}{\sqrt{c_1}} f_1',$$
$$\Gamma_1 : S^+ \to \mathbb{C}, \quad \Gamma_1(\{f,f'\}) = -\sqrt{c_1} f_1 + \sqrt{c_2} f_2 + \sqrt{c_3} f_3 - \tfrac{1}{\sqrt{c_3}}(5f_3 - f_3'),$$

is a boundary triplet for S^+. For $\{f,f'\} \in \ker\Gamma_0 =: A$ we have

$$\tfrac{-1}{\sqrt{c_1}} f_1' = \tfrac{1}{\sqrt{c_2}}(f_2 - f_2') = \tfrac{1}{\sqrt{c_3}}(5f_3 - f_3') = 0,$$

i.e. $f_1' = 0$, $f_2' = f_2$ and $f_3' = 5f_3$. Thus

$$A = \{ \{f,f'\} \in \mathbb{C}^3 \times \mathbb{C}^3 \,|\, f = (f_1,f_2,f_3)^\top, f' = (0, f_2, 5f_3)^\top \}$$

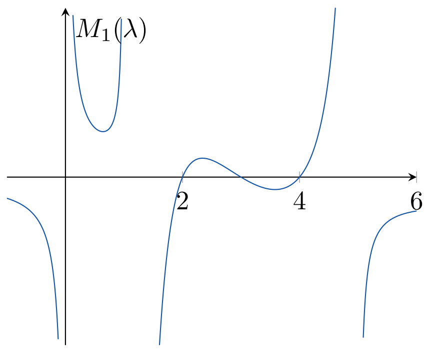

Figure 5.2: Schematic plot of the function $M_1(\lambda) = -\frac{(\lambda-2)(\lambda-3)(\lambda-4)}{\lambda(\lambda-1)(\lambda-5)}$.

with the matrix representation

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Now let $\{f, f'\} \in B := \ker \Gamma_1$. Then $\{f, f'\} \in S^+$ and we obtain the three equations

$$\begin{aligned} \tfrac{1}{\sqrt{c_3}}(5f_3 - f_3') &= -\sqrt{c_1}f_1 + \sqrt{c_2}f_2 + \sqrt{c_3}f_3, \\ \tfrac{1}{\sqrt{c_2}}(f_2 - f_2') &= -\sqrt{c_1}f_1 + \sqrt{c_2}f_2 + \sqrt{c_3}f_3, \\ -\tfrac{1}{\sqrt{c_1}}f_1' &= -\sqrt{c_1}f_1 + \sqrt{c_2}f_2 + \sqrt{c_3}f_3, \end{aligned}$$

which yields

$$\begin{aligned} f_1' &= c_1 f_1 - \sqrt{c_1c_2}f_2 - \sqrt{c_1c_3}f_3, \\ f_2' &= \sqrt{c_1c_2}f_1 + (1-c_2)f_2 - \sqrt{c_2c_3}f_3, \quad \text{and} \\ f_3' &= \sqrt{c_1c_3}f_1 - \sqrt{c_2c_3}f_2(5-c_3)f_3. \end{aligned}$$

Hence we see

$$B = \begin{pmatrix} \frac{24}{5} & -\frac{6}{\sqrt{5}} & -\frac{6}{5} \\ \frac{6}{\sqrt{5}} & -\frac{1}{2} & -\frac{3}{\sqrt{20}} \\ \frac{6}{5} & -\frac{3}{\sqrt{20}} & \frac{47}{10} \end{pmatrix}.$$

Then A and B are selfadjoint in the Krein space $(\mathbb{C}^3, [\cdot,\cdot])$ and differ by a rank one matrix. Clearly $\sigma(A) = \{0, 1, 5\}$ coincides with the poles of M_1 and the zeros of M_1 coincide with $\sigma(B) = \{2, 3, 4\}$. We also mention that A is non-negative and it can be checked that B has one negative square. Obviously the matrix B has three eigenvalues in the interval $(1, 5)$ whereas A has no eigenvalues in $(1, 5)$, cf. the upper estimate in Theorem 5.12(i) with $\kappa_B = 1$. Moreover, in $(-1, 2)$ are no eigenvalues of B whereas A has two eigenvalues there, cf. the lower estimate in Theorem 5.12(ii). Similarly, any sufficiently small interval containing a positive pole of M_1 is an example for the lower estimate in Theorem 5.12(i).

Example 5.16. As a second example consider the function

$$M_2(\lambda) = -\frac{(\lambda+1)(\lambda-1)(\lambda-3)}{(\lambda+2)(\lambda-2)(\lambda-4)},$$

cf. Figure 5.3. With

$$\begin{aligned} M_2(\lambda) &= \frac{5}{8(\lambda+2)} - \frac{3}{8(\lambda-2)} - \frac{5}{4(\lambda-4)} - 1 \quad \text{and} \\ -\frac{1}{M_2(\lambda)} &= \frac{15}{8(\lambda+1)} - \frac{9}{4(\lambda-1)} - \frac{5}{8(\lambda-3)} + 1 \end{aligned}$$

we see that M_2 and $-M_2^{-1}$ belong to $\mathcal{D}_0$ by Theorem 5.8. We equip $\mathbb{C}^3$ with the indefinite inner product (5.22). As in Example 5.15 we construct a realisation of M_2 with the methods of Section 2.4: Set

$$c_1 = \frac{5}{8}, \quad c_2 = \frac{3}{8}, \quad \text{and} \quad c_3 = \frac{5}{4},$$

and

$$\begin{aligned} S := \Big\{ \{f, f'\} \in \mathbb{C}^3 \times \mathbb{C}^3 \,\Big|\, &\tfrac{1}{\sqrt{c_1}}(-2f_1 - f_1') = \tfrac{1}{\sqrt{c_2}}(2f_2 - f_2') = \tfrac{1}{\sqrt{c_3}}(4f_3 - f_3') = 0, \\ &-\sqrt{c_1}f_1 + \sqrt{c_2}f_2 + \sqrt{c_3}f_3 - \tfrac{1}{\sqrt{c_3}}(4f_3 - f_3') = 0 \Big\}. \end{aligned}$$

Then

$$S^+ = \Big\{ \{f, f'\} \in \mathbb{C}^3 \times \mathbb{C}^3 \,\Big|\, \tfrac{1}{\sqrt{c_1}}(-2f_1 - f_1') = \tfrac{1}{\sqrt{c_2}}(2f_2 - f_2') = \tfrac{1}{\sqrt{c_3}}(4f_3 - f_3') \Big\}$$

and the triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ given by

$$\begin{aligned} &\Gamma_0 : S^+ \to \mathbb{C}, \quad \Gamma_0(\{f, f'\}) = \tfrac{1}{\sqrt{c_1}}(-2f_1 - f_1'), \\ &\Gamma_1 : S^+ \to \mathbb{C}, \quad \Gamma_1(\{f, f'\}) = -\sqrt{c_1}f_1 + \sqrt{c_2}f_2 + \sqrt{c_3}f_3 - \tfrac{1}{\sqrt{c_3}}(4f_3 - f_3'), \end{aligned}$$

is a boundary triplet for S^+. Calculating $A := \ker \Gamma_0$ and $B := \ker \Gamma_1$ we obtain the selfadjoint matrices

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -\frac{11}{8} & -\frac{\sqrt{15}}{8} & -\frac{5}{4\sqrt{2}} \\ \frac{\sqrt{15}}{8} & \frac{13}{8} & -\sqrt{\frac{15}{32}} \\ \frac{5}{4\sqrt{2}} & -\sqrt{\frac{15}{32}} & \frac{11}{4} \end{pmatrix}$$

It can be checked that in fact $A - B$ is a rank one matrix, $\kappa_B = 0$, and that $\sigma(A) = \{-2, 2, 4\}$ and $\sigma(B) = \{-1, 1, 3\}$ are the poles and zeros of M_2, respectively. The matrix B has two eigenvalues in the interval $(-2, 2)$ whereas A has no eigenvalue in $(-2, 2)$, which is the upper estimate in Theorem 5.12(ii) with $\kappa_B = 0$. Similarly, any sufficiently small interval

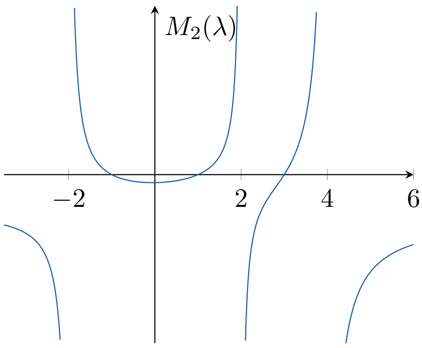

Figure 5.3: Schematic plot of the function $M_2(\lambda) = -\frac{(\lambda+1)(\lambda-1)(\lambda-3)}{(\lambda+2)(\lambda-2)(\lambda-4)}$.

containing a zero of M_2 is an example for the upper estimate in Theorem 5.12(i) with $\kappa_B = 0$.

Example 5.17. Consider the function

$$M_3(\lambda) = -\frac{(\lambda+1)(\lambda-1)(\lambda-2)(\lambda-3)}{(\lambda+2)\lambda^2(\lambda-4)}.$$

Here we have

$$M_3(\lambda) = -\frac{3}{4\lambda^2} + \frac{13}{16\lambda} + \frac{5}{2(\lambda+2)} - \frac{5}{16(\lambda-4)} - 1 \quad \text{and}$$
$$-\frac{1}{M_3(\lambda)} = \frac{5}{24(\lambda+1)} - \frac{9}{4(\lambda-1)} + \frac{32}{3(\lambda-2)} - \frac{45}{8(\lambda-3)} + 1.$$

With Theorem 5.8 we see $M_3 \in \mathcal{D}_0$ and $-M_3^{-1} \in \mathcal{D}_1$ (note that 0 is no generalized pole of non-positive type of $\lambda \mapsto \lambda M_3(\lambda)$ and 2 is a generalized pole of non-positive type with multiplicity one of $-M_3^{-1}$).

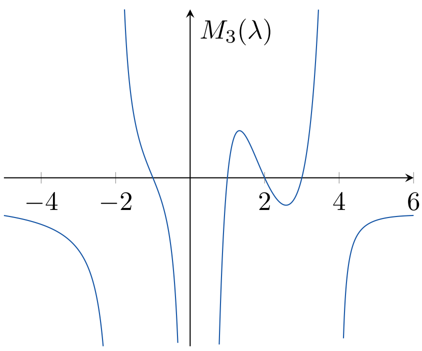

Figure 5.4: Schematic plot of the function $M_3(\lambda) = -\frac{(\lambda+1)(\lambda-1)(\lambda-2)(\lambda-3)}{(\lambda+2)\lambda^2(\lambda-4)}$.

We equip $\mathbb{C}^5$ with the indefinite inner product

$$[\cdot,\cdot] := \langle J\cdot,\cdot\rangle, \quad J = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix},$$

and construct a realisation of M_3 as in Examples 5.15 and 5.16. For this, set

$$c_1 = \frac{3}{4}, \quad c_3 = \frac{13}{16}, \quad c_4 = \frac{5}{2}, \quad \text{and} \quad c_5 = \frac{5}{16},$$

and consider

$$\begin{aligned} S := \Big\{ \{f, f'\} \in \mathbb{C}^5 \times \mathbb{C}^5 \;\Big|\; & f_1' = f_2,\ \tfrac{-1}{\sqrt{c_1}} f_2' = \tfrac{-1}{\sqrt{c_3}} f_3' = \tfrac{-1}{\sqrt{c_4}}(2f_4 + f_4') = \tfrac{1}{\sqrt{c_5}}(4f_5 - f_5') = 0, \\ & \sqrt{c_1} f_1 - \sqrt{c_3} f_3 - \sqrt{c_4} f_4 + \sqrt{c_5} f_5 - \tfrac{1}{\sqrt{c_5}}(4f_5 - f_5') = 0 \Big\}. \end{aligned}$$

Then

$$S^+ = \Big\{ \{f, f'\} \in \mathbb{C}^3 \times \mathbb{C}^3 \;\Big|\; f_1' = f_2,\ \tfrac{-1}{\sqrt{c_1}} f_2' = \tfrac{-1}{\sqrt{c_3}} f_3' = \tfrac{-1}{\sqrt{c_4}}(2f_4 + f_4') = \tfrac{1}{\sqrt{c_5}}(4f_5 - f_5') \Big\}$$

and the triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ given by

$$\begin{aligned} &\Gamma_0 : S^+ \to \mathbb{C}, \quad \Gamma_0(\{f, f'\}) = -\tfrac{1}{\sqrt{c_1}} f_2', \\ &\Gamma_1 : S^+ \to \mathbb{C}, \quad \Gamma_1(\{f, f'\}) = \sqrt{c_1} f_1 - \sqrt{c_3} f_3 - \sqrt{c_4} f_4 + \sqrt{c_5} f_5 - \tfrac{1}{\sqrt{c_5}}(4f_5 - f_5'), \end{aligned}$$

is a boundary triplet for S^+. Calculating $A := \ker \Gamma_0$ and $B := \ker \Gamma_1$ we see

$$A = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & 0 & & \\ & & & -2 & \\ & & & & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{3}{4} & 0 & \frac{\sqrt{39}}{8} & \sqrt{\frac{15}{8}} & -\frac{\sqrt{15}}{8} \\ -\frac{\sqrt{39}}{8} & 0 & \frac{13}{16} & \sqrt{\frac{65}{32}} & -\frac{\sqrt{65}}{16} \\ -\sqrt{\frac{15}{8}} & 0 & \sqrt{\frac{65}{32}} & \frac{1}{2} & -\frac{5}{\sqrt{32}} \\ -\frac{\sqrt{15}}{8} & 0 & \frac{\sqrt{65}}{16} & \frac{5}{\sqrt{32}} & \frac{59}{16} \end{pmatrix}.$$

Then $A - B$ is a rank one matrix, $\kappa_B = 1$, $\sigma(A) = \{-2, 0, 4\}$, and $\sigma(B) = \{-1, 0, 1, 2, 3\}$. In the interval $(-2, 4)$ the matrix B has 5 eigenvalues whereas A has one eigenvalue there and $n_{A,B}((-2, 4)) = 1$, cf. the upper estimate in Theorem 5.12(ii) with $\kappa_B = 1$.

5.3 Total Multiplicity of Eigenvalues in an Interval

In the following we provide in Theorem 5.18 a variant of Theorem 5.12, where the *total multiplicity $m_B(I)$ of the eigenvalues of B in I* is estimated by the *total multiplicity $m_A(I)$ of the eigenvalues of A in I*, cf. [17].

Theorem 5.18. *Let A, B, and I be as in Assumption* I *and assume in addition that A is non-negative and $m_A(I) < \infty$. Then the following estimates hold.*

(i) *If* $0 \notin I$ *then*

$$m_A(I) - 1 \le m_B(I) \le m_A(I) + \begin{cases} 1 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(ii) *If* $0 \in I$ *and* $0 \notin \sigma_{\mathrm{p}}(A)$ *then*

$$m_A(I) - 2 \le m_B(I) \le m_A(I) + \begin{cases} 2 & \text{if } \kappa_B = 0, \\ 3 & \text{if } \kappa_B = 1. \end{cases}$$

(iii) *If* $0 \in I$ *and* $0 \in \sigma_{\mathrm{p}}(A)$ *then*

$$m_A(I) - 4 \le m_B(I) \le m_A(I) + \begin{cases} 4 & \text{if } \kappa_B = 0, \\ 6 & \text{if } \kappa_B = 1. \end{cases}$$

Moreover, $m_A(I) = \infty$ *if and only if* $m_B(I) = \infty$*.*

Remark 5.19. It follows immediately from Corollary 5.14 that the estimates in Theorem 5.18 (i) and (ii) are sharp. It is not clear if estimate (iii) is sharp as well.

Proof of Theorem 5.18. The proof of Theorem 5.18 uses Corollary 5.14 and is done in eleven steps. We decompose the space $\mathcal{K}$ into the spectral subspace related to the common eigenvalues of A and B and its $[\cdot,\cdot]$-orthogonal complement. Then Corollary 5.14 can be applied to the restrictions of A and B to this $[\cdot,\cdot]$-orthogonal complement and we prove the estimates in (i), (ii), and (iii).

Step 1. Decomposition of $\mathcal{K}$ for $\mathbf{0 \notin I}$. Let us assume that $I \subseteq (0,\infty)$. The spectral subspace of A corresponding to I is an $m_A(I)$-dimensional Hilbert space by Corollary 1.14(i). The subspace $\mathcal{E}_+$ spanned by the eigenvectors of the (possibly non-densely defined) symmetric operator $S = A \cap B$ in I is invariant for S, and hence for A and B. As $\mathcal{E}_+$ is a subset of the spectral subspace of A corresponding to I, the space $(\mathcal{E}_+, [\cdot,\cdot])$ is a (finite-dimensional) Hilbert space. Denote the restriction of S to $\mathcal{E}_+$ by S_+. With respect to the decomposition $\mathcal{K} = \mathcal{E}_+ [\dot{+}] \mathcal{E}_+^{[\perp]}$ we have

$$S = \begin{pmatrix} S_+ & 0 \\ 0 & S' \end{pmatrix}, \quad A = \begin{pmatrix} S_+ & 0 \\ 0 & A' \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} S_+ & 0 \\ 0 & B' \end{pmatrix},$$

with S' symmetric, $\sigma_{\mathrm{p}}(S') \cap I = \emptyset$, and A' and B' selfadjoint in the Krein space $(\mathcal{E}_+^{[\perp]}, [\cdot,\cdot])$. Therefore

$$m_A(I) = m_{S_+}(I) + m_{A'}(I) \quad \text{and} \quad m_B(I) = m_{S_+}(I) + m_{B'}(I). \tag{5.23}$$

We claim that A' and B' satisfy the assumptions in Corollary 5.14. Indeed, it is easy to see that A', B', and I satisfy Assumption I and since A is non-negative in the Krein space $\mathcal{K}$ the operator A' is non-negative in the Krein space $\mathcal{E}_+^{[\perp]}$. Furthermore, as $\sigma_{\mathrm{p}}(S') \cap I =$

$\emptyset$ and all eigenvalues of A' in I are in $\sigma_{++}(A')$ by Corollary 1.14(i), we conclude from Proposition 2.16(i) that

$$\sigma_{\mathrm{p}}(A') \cap \sigma_{\mathrm{p}}(B') \cap I = \emptyset. \tag{5.24}$$

Step 2. Lower estimate in (i). As $I \subseteq (0, \infty)$, all eigenvalues of the non-negative operator A' in I are of positive type and belong to $\rho(B')$. According to Theorem 2.12(ii) each of these eigenvalues is of multiplicity one and therefore

$$n_{A'}(I) = m_{A'}(I). \tag{5.25}$$

As $n_{B'}(I) \leq m_{B'}(I)$, Corollary 5.14(i) together with (5.23) imply the estimate

$$m_A(I) - 1 \leq m_B(I). \tag{5.26}$$

Step 3. Upper estimate in (i) if $\kappa_{\mathbf{B}} = \mathbf{0}$. The estimate follows immediately from (5.26) by interchanging the roles of A and B.

Step 4. Upper estimate in (i) if $\kappa_{\mathbf{B}} = \mathbf{1}$. In this case $\kappa_{B'} = 1$ and by Corollary 1.14(ii) there is at most one eigenvalue μ of B' in I which is not of positive type. If μ is of negative type it has multiplicity one, cf. Theorem 2.12(i). All other eigenvalues of B' in I are of positive type, belong to $\rho(A')$ and hence have multiplicity one according to Corollary 1.14(ii) and Theorem 2.12(i). Therefore $n_{B'}(I) = m_{B'}(I)$ and as $n_{A'}(I) \leq m_{A'}(I)$, Corollary 5.14(i) together with (5.23) imply the estimate

$$m_B(I) \leq m_A(I) + 3. \tag{5.27}$$

It remains to show (5.27) in the case that $\mu \in \sigma_{\mathrm{p}}(B') \cap I$ is not of positive and not of negative type, that is, there exists a neutral eigenvector x_0. Then by Lemma 2.13 $\dim \ker(B' - \mu) = 1$ and the multiplicity of μ is larger than one. On the other hand it follows from Theorem 1.13(ii) that the multiplicity of μ is at most 3. We discuss the cases $\dim \mathcal{L}_\mu(B') = 2$ and $\dim \mathcal{L}_\mu(B') = 3$ separately.

If $\dim \mathcal{L}_\mu(B') = 3$ then there exists a Jordan chain $\{x_0, x_1, x_2\}$ of B' at μ of length 3, and (2.9) implies $M'_{A'}(\mu) = 0$ and

$$M''_{A'}(\mu) = 2[x_1, x_0] = 2[(B' - \mu)x_2, x_0] = 2[x_2, (B' - \mu)x_0] = 0. \tag{5.28}$$

By Proposition 5.10(iii) we have $M_{B'} \in \mathcal{D}_1$ and Proposition 5.10(ii) yields

$$M'''_{A'}(\mu) > 0. \tag{5.29}$$

As in Step 4 in the proof of Theorem 5.12 the set $I \cap \rho(A')$ consists of $n_{A'}(I) + 1 = m_{A'}(I) + 1$ open subintervals I_k. We have $\mu \in \rho(A')$ (see (5.24)) and hence $\mu \in I_j$ for some j with $1 \leq j \leq m_{A'}(I) + 1$. Since all other eigenvalues of B' in $I \cap \rho(A')$ belong to $\sigma_{++}(B')$ it follows from Lemma 2.13(i) that the derivative of $M_{A'}$ in such an eigenvalue is positive. This together with (5.29) shows that except for μ there is no other eigenvalue of B' in I_j. Moreover in each of the subintervals I_k, $1 \leq k \leq m_{A'}(I) + 1$, $k \neq j$, there is at most one eigenvalue of B'. Summing up we have

$$m_{B'}(I) = n_{B'}(I) + 2 \quad \text{and} \quad n_{B'}(I) \le n_{A'}(I) + 1.$$

Together with (5.23) and (5.25) the estimate (5.27) follows if the multiplicity of μ is 3.

It remains to consider the case $\dim \mathcal{L}_\mu(B') = 2$. Relation (2.9) implies $M'_{A'}(\mu) = [x_0, x_0] = 0$. If $M''_{A'}(\mu) = 0$ then a similar reasoning as above implies (5.29) and the estimate (5.27) follows in the same way. If $M''_{A'}(\mu) \neq 0$ then we consider again the open subintervals I_k from above, $1 \le k \le m_{A'}(I) + 1$, and for some subinterval I_j with $1 \le j \le m_{A'}(I) + 1$ we have $\mu \in I_j$. Again, by Lemma 2.13(i), the derivative of $M_{A'}$ is positive in all eigenvalues except in μ. Hence in each I_k, $k \neq j$, there is at most one eigenvalue of B'. In I_j the eigenvalue μ has multiplicity 2 and Lemma 2.14 yields that there is precisely one more eigenvalue of B' (with multiplicity one) in I_j. This implies

$$m_{B'}(I) = n_{B'}(I) + 1 \quad \text{and} \quad n_{B'}(I) \le n_{A'}(I) + 2.$$

With (5.23) and (5.25) the upper estimate in (i) with $\kappa_B = 1$ follows.

Step 5. Lower estimate in (ii) and (iii). If $0 \in I$ we apply the lower estimate in (i) to the intervals $I_+ = I \cap (0,\infty)$ and $I_- = I \cap (-\infty, 0)$ separately. Taking into account the assumption $0 \notin \sigma_p(A)$ we obtain the lower estimate in (ii). If $0 \in \sigma_p(A)$ we obtain

$$\begin{aligned} m_A(I) - 2 &= m_A(I_+) - 1 + m_A(I_-) - 1 + m_A(\{0\}) \\ &\le m_B(I_+) + m_B(I_-) + m_B(\{0\}) - m_B(\{0\}) + m_A(\{0\}) \\ &\le m_B(I) + |m_A(\{0\}) - m_B(\{0\})| \end{aligned}$$

and the lower estimate in (iii) follows from Theorem 4.6.

Step 6. Decomposition of $\mathcal{K}$ if $\mathbf{0} \in \mathbf{I}$. As in Step 1 the spectral subspace of A corresponding to $I_+ = I \cap (0,\infty)$ ($I_- = I \cap (-\infty, 0)$) is a Hilbert space (anti-Hilbert space, respectively), cf. Corollary 1.14(i). The subspace $\mathcal{E}_+$ ($\mathcal{E}_-$) spanned by the eigenvectors of $S = A \cap B$ in I_+ (I_-) is a subset of the spectral subspace of A corresponding to I_+ (I_-, respectively), and the space $\mathcal{E} := \mathcal{E}_+[\dot{+}]\mathcal{E}_-$ is a Krein space. Denote the restriction of S to $\mathcal{E}$ by $S_\mathcal{E}$. With respect to the decomposition $\mathcal{K} = \mathcal{E}[\dot{+}]\mathcal{E}^{[\perp]}$ we have

$$S = \begin{pmatrix} S_\mathcal{E} & 0 \\ 0 & S' \end{pmatrix}, \quad A = \begin{pmatrix} S_\mathcal{E} & 0 \\ 0 & A' \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} S_\mathcal{E} & 0 \\ 0 & B' \end{pmatrix},$$

with S' symmetric, $\sigma_p(S') \cap I \subseteq \{0\}$, A' non-negative, and B' selfadjoint in the Krein space $(\mathcal{E}^{[\perp]}, [\cdot,\cdot])$. Again A', B', and I satisfy Assumption I and, as in (5.23), we have

$$m_A(I) = m_{S_\mathcal{E}}(I) + m_{A'}(I) \quad \text{and} \quad m_B(I) = m_{S_\mathcal{E}}(I) + m_{B'}(I). \tag{5.30}$$

If $0 \notin \sigma_p(A)$ then $0 \notin \sigma_p(A')$ and we conclude from Proposition 2.16(i) in the same way as in Step 1 that

$$\sigma_p(A') \cap \sigma_p(B') \cap I = \emptyset. \tag{5.31}$$

Step 7. Upper estimate in (ii) if $\kappa_\mathbf{B} = \mathbf{0}$. In the case $0 \notin \sigma_\mathrm{p}(B)$ the upper estimate in (ii) for $\kappa_B = 0$ follows immediately from the lower estimate in Step 5 by interchanging the roles of A and B.

Hence we consider the case $0 \in \sigma_\mathrm{p}(B)$. Then we also have $0 \in \sigma_\mathrm{p}(B')$. As $0 \notin \sigma_\mathrm{p}(A')$ Theorem 2.12(ii) implies $n_{A'}(I) = m_{A'}(I)$ also for an interval which contains 0. The set $I \cap \rho(A')$ consists of $n_{A'}(I) + 1 = m_{A'}(I) + 1$ open subintervals I_k. We have $0 \in \rho(A')$ and hence $0 \in I_j$ for some j with $1 \leq j \leq m_{A'}(I) + 1$. As B and B' are non-negative operators all eigenvalues of B' in I_+ (I_-) belong to $\sigma_{++}(B')$ ($\sigma_{--}(B')$, respectively). It follows from Lemma 2.13(i)–(ii) and (5.31) that the derivative of $M_{A'}$ in eigenvalues of B' in I_+ (I_-) is positive (negative, respectively) and the multiplicity of these eigenvalues is one. We estimate the multiplicity of the eigenvalues of B' in I_j. Since $0 \in \sigma_\mathrm{p}(B') \cap \rho(A')$ we have $M_{A'}(0) = 0$ and by Lemma 5.11(iii) the point 0 is a zero of $M_{A'}$ of at most order two. If it is of order two, Lemma 5.11(iii) and the above reasoning imply that 0 is the only zero in I_j. As B' is a non-negative operator, the (algebraic) multiplicity of the eigenvalue 0 is at most two. If 0 is a zero of $M_{A'}$ of order one then the sign properties of $M'_{A'}$ at the other zeros yield that there is at most one more eigenvalue of B' in I_j. As a consequence of Lemma 2.13(i)–(ii) the multiplicities of these two eigenvalues in I_j are both one. Therefore in both cases we have

$$m_{B'}(I) \leq m_{A'}(I) + 2.$$

Together with (5.30) the upper estimate in (ii) in the case $\kappa_B = 0$ is shown.

Step 8. Upper estimate in (ii) if $\kappa_\mathbf{B} = \mathbf{1}$. We again make use of the open subintervals I_k from Step 7 such that $0 \in I_j$. We proceed in a similar way as in Step 5 of the proof of Theorem 5.12. By Proposition 5.10 the function $M_{A'}$ has at most one zero $\mu \in I_{k_0}$ in a subinterval I_{k_0}, $k_0 \neq j$, with $M'_{A'}(\mu) \leq 0$ if $\mu > 0$ or $M'_{A'}(\mu) \geq 0$ if $\mu < 0$. If $M_{A'}$ has such an exceptional zero, then by Proposition 5.10(iii) $M_{B'} \in \mathcal{D}_1$ and, hence, $M_{A'}(0) > 0$ by Lemma 5.11(ii). Thus $M_{A'}$ has no zero in I_j and therefore B' has no eigenvalue in I_j. As in Step 4 of the proof of Theorem 5.12 it follows that the total multiplicity of the eigenvalues of B' in I_{k_0} is at most three. Moreover, in the other subintervals I_k, $k \neq k_0$, $k \neq j$, B' has at most one eigenvalue of multiplicity one. This yields the upper estimate in (ii).

It remains to discuss the case that $M_{A'}$ has at most one zero in each of the subintervals I_k, $k \neq j$, with positive (negative) derivative at these zeros if they are in $I_k \subseteq (0, \infty)$ ($I_k \subseteq (-\infty, 0)$, respectively). We distinguish in this situation the cases $M_{A'}(0) > 0$, $M_{A'}(0) = 0$, and $M_{A'}(0) < 0$.

Observe that in the first case there is no zero of $M_{A'}$ of third order in I_j (Proposition 5.10(ii)) and there may appear either one zero of $M_{A'}$ of second order or two zeros of order one in I_j, cf. Proposition 5.10. Hence we have either one eigenvalue of B' of multiplicity two (cf. (5.28) in Step 4) or two eigenvalues of multiplicity one. If $M_{A'}(0) = 0$ then $M_{B'} \in \mathcal{D}_0$ by Lemma 5.11(ii) and 0 is a zero of at most second order by Lemma 5.11(iii). If 0 is a zero of second order then $M''_{A'}(0) > 0$, there are no other zeros of $M_{A'}$ in I_j (Proposition 5.10(i)), and therefore 0 is an eigenvalue of B' of multiplicity two (cf. (5.28) in Step 4). If 0 is a zero of first order there is at most one other zero in I_j of multiplicity one (Proposition 5.10(i)); thus the total multiplicity of the eigenvalues of B' in I_j is at most two. If $M_{A'}(0) < 0$ then again $M_{B'} \in \mathcal{D}_0$ by Lemma 5.11(ii) and it follows from Proposition 5.10(i) that $M_{A'}$ has at most two zeros of first order in I_j. Again, the total multiplicity of the eigenvalues of B' in I_j is at most two and the upper estimate in (ii) follows.

Step 9. Upper estimate in (iii) if $\kappa_\mathbf{B} = \mathbf{0}$. The upper estimate in (iii) for $\kappa_B = 0$ follows from Theorem 4.6 and from the upper estimate in (i) applied to the intervals $I_+ = I \cap (0, \infty)$ and $I_- = I \cap (-\infty, 0)$ separately.

Step 10. Upper estimate in (iii) if $\kappa_\mathbf{B} = \mathbf{1}$. From Proposition 2.16(i) we conclude

$$\sigma_\mathrm{p}(A') \cap \sigma_\mathrm{p}(B') \cap (I_- \cup I_+) = \emptyset$$

and Theorem 2.12(ii) implies

$$n_{A'}(I_- \cup I_+) = m_{A'}(I_- \cup I_+).$$

By Proposition 5.10(ii) the function $M_{A'}$ has at most one zero μ in I_+ (I_-) with $M'_{A'}(\mu) \leq 0$ ($M'_{A'}(\mu) \geq 0$, respectively). For simplicity, we assume that $M'_{A'}$ has such an exceptional zero μ in I_-. As in Step 4 of the proof of Theorem 5.12 it follows that the total multiplicity of the eigenvalues of B' in I_- exceeds the total multiplicity of the eigenvalues of A' in I_- by at most 3, whereas in I_+ it exceeds by at most 1, hence

$$m_{B'}(I_- \cup I_+) \leq m_{A'}(I_- \cup I_+) + 4.$$

Together with Theorem 4.6 we obtain

$$m_{B'}(I) = m_{B'}(I_- \cup I_+) + m_{B'}(\{0\}) \leq m_{A'}(I_- \cup I_+) + 4 + m_{A'}(\{0\}) + 2 = m_{A'}(I) + 6$$

and, together with (5.30) the upper estimate in (iii) is shown.

Step 11. Proof of $\mathbf{m_A(I)} = \infty$ if and only if $\mathbf{m_B(I)} = \infty$. If $m_A(I) = \infty$ then either $n_A(I) = \infty$ in which case the assertion follows from Theorem 5.12, or $n_A(I) < \infty$ in which case there exists at least one eigenvalue μ of A with infinite multiplicity. With Proposition 1.11 we see $\dim \ker(A - \mu) = \infty$. Then $\dim \ker(B - \mu) = \infty$ by Theorem 3.1(i), which yields $m_B(I) = \infty$. Conversely, if $m_B(I) = \infty$ an analogous reasoning implies $m_A(I) = \infty$. $\square$

From Step 8 of the proof above we immediately obtain the following corollary.

Corollary 5.20. *Let A, B, and I be as in Theorem 5.18. Assume that $0 \notin \sigma_p(A)$ and that the minimal positive (maximal negative, respectively) eigenvalue λ_+ (λ_-) of A exists. If $\kappa_B = 1$ then there exists at most one pair of consecutive eigenvalues $\lambda_1 < \lambda_2$ of A in $I \backslash (\lambda_-, \lambda_+)$ such that $m_B((\lambda_1, \lambda_2)) \geq 2$. If such a pair exists then B has no eigenvalues in (λ_-, λ_+). If no such pair λ_1, λ_2 exists then $m_B((\lambda_-, \lambda_+)) \leq 2$.*

6. Conclusions

As a conclusion of this thesis we combine in this chapter the results of Chapters 4 and 5. We consider the cases

(1) B is non-negative and $0 \notin I$, (2) B is non-negative and $0 \in I$,
(3) B has one negative square and $0 \notin I$, (4) B has one negative square and $0 \in I$.

For each of these cases we describe the spectrum of B in I. For convenience we repeat the assumptions imposed on A and B.

Assumption I. Let A and B be selfadjoint operators in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that (2.5) holds for some $\lambda_0 \in \rho(A) \cap \rho(B)$ with functions M_A and M_B as in Proposition 2.10. Let $I \subseteq \mathbb{R}$ be an open interval and assume that $\rho(B) \cap I \neq \emptyset$ and $\sigma(A) \cap I$ consists only of isolated eigenvalues which are poles of the resolvent of A.

In the following proposition we collect the results from Sections 3.2 and 4.1 describing the Jordan structure of B at 0, i.e. the number and length of the linearly independent Jordan chains of B at 0 compared with the Jordan chains of A at 0.

Proposition 6.1. *Let A, B, and I be as in Assumption* I. *Assume that A is non-negative and let $0 \in I$ with $\dim \mathfrak{L}_0(A) < \infty$. Then the following holds.*

(i) *A has Jordan chains of length at most 2 at 0 and B has Jordan chains of length at most 4 at 0. If in addition B is non-negative then B has Jordan chains of length at most 2 at 0.*

(ii) *The number of linearly independent Jordan chains of B at 0 is bounded:*

$$|\dim \ker A - \dim \ker B| \leq 1, \quad \left|\dim\left(kerA^2/\ker A\right) - \dim\left(\ker B^2/\ker B\right)\right| \leq 1,$$
$$\dim\left(\ker B^3/\ker B^2\right) \leq 1, \qquad \textit{and} \quad \dim\left(\ker B^4/\ker B^3\right) \leq 1.$$

(iii) *For the dimensions of the root subspaces of A and B at 0 we have*

$$|\dim \mathfrak{L}_0(A) - \dim \mathfrak{L}_0(B)| \leq 2.$$

(iv) *B has at most one Jordan chain of length ≥ 3 at 0. If such a Jordan chain exists then B has one negative square,* $\ker B$ *is contained in* $\ker A$*, and*

$$\dim \mathfrak{L}_0(A) = \dim \ker A^2 \geq \dim \ker B^2.$$

Moreover, A does not have more linearly independent Jordan chains of length 2 at 0 than B and if A and B have equally many linearly independent Jordan chains of length 2 at 0, then $\ker B$ and $\ker A$ coincide.

(v) *If all Jordan chains of B at 0 are of length at most 2 and the dimensions of $\ker B$ and $\ker A$ coincide, then $\dim \mathfrak{L}_0(B) = \dim \mathfrak{L}_0(A)$.*

Proof. (i) and (ii) follow from Lemma 4.1 and Theorem 3.1 and (iii) holds by Theorem 4.6. To prove (iv) note that by Remark 4.2 B has at most one Jordan chain of length ≥ 3 at 0. Let there exist such a chain. Then the remaining statements are consequences of Propositions 4.3 and 4.5. Finally, (v) follows from Proposition 3.10(ii). □

We describe the spectrum of B in the Cases (1) – (4).

Theorem 6.2. *Let A, B, and I be as in Assumption I. Assume in addition that $m_A(I) < \infty$ and*

$$A \text{ is non-negative}, \quad B \text{ is non-negative}, \quad \text{and} \quad 0 \notin I.$$

Then the following hold.

(i) *For the number of distinct eigenvalues of B in I we have*

$$n_A(I) - n_{A,B}(I) - 1 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + 1.$$

(ii) *For the total multiplicity of eigenvalues of B in I we have*

$$m_A(I) - 1 \leq m_B(I) \leq m_A(I) + 1.$$

(iii) *For every $\mu \in I$ the dimension of $\ker(B-\mu)$ is either one less, equal to, or one greater than the dimension of $\ker(A-\mu)$ and A and B have no Jordan chain of length ≥ 2 at μ.*

(iv) *Between two consecutive eigenvalues of A in I there is at most one eigenvalue μ of B and if it exists then $\dim \mathfrak{L}_\mu(B) = \dim\ker(B-\mu) = 1$.*

Proof. (i) and (ii) follow directly from Theorems 5.12(i) and 5.18(i), respectively, and (iii) follows from Theorem 3.1 since A and B are non-negative and thus have no Jordan chains of length ≥ 2 at $\mu \neq 0$, cf. Remark 1.15. Let $0 < \lambda_1 < \lambda_2$ be two consecutive eigenvalues of A in I. Then $n_A((\lambda_1, \lambda_2)) = n_{A,B}((\lambda_1, \lambda_2)) = 0$ and the first statement in (iv) follows from Theorem 5.12(i) applied to (λ_1, λ_2). Finally $\dim \mathfrak{L}_\mu(B) = 1$ holds by Theorem 3.1 since $\mu \in \rho(A)$ and B is non-negative. □

Theorem 6.3. *Let A, B, and I be as in Assumption I. Assume in addition that $m_A(I) < \infty$ and*

$$A \text{ is non-negative}, \quad B \text{ is non-negative}, \quad \text{and} \quad 0 \in I.$$

Then the following hold.

(i) *For the number of distinct eigenvalues of B in I we have*

$$n_A(I) - n_{A,B}(I) - 2 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + 2.$$

(ii) *For the total multiplicity of eigenvalues of B in I we have*

$$\begin{aligned} m_A(I) - 2 \leq m_B(I) \leq m_A(I) + 2 \quad &\text{if } 0 \notin \sigma_p(A), \\ m_A(I) - 4 \leq m_B(I) \leq m_A(I) + 4 \quad &\text{if } 0 \in \sigma_p(A). \end{aligned}$$

(iii) *For every $\mu \in I \backslash \{0\}$ the dimension of $\ker(B - \mu)$ is either one less, equal to, or one greater than the dimension of $\ker(A - \mu)$ and B has no Jordan chain of length ≥ 2 at μ.*

(iv) *The Jordan structure of B at 0, i.e. the number and length of all Jordan chains of B at 0 compared with the Jordan chains of A at 0 is described in Proposition 6.1. In particular, there are 7 possibilities for the Jordan structure of B at 0 which are given in Table 4.1, Cases $1, 4, 7, 13, 18, 19,$ and 20.*

(v) *If $0 \in \sigma_p(A)$ then between two consecutive eigenvalues of A in I there is at most one eigenvalue μ of B. If $0 \in \rho(A)$ then between two consecutive eigenvalues $\lambda_1 < \lambda_2$ of A in I such that $0 \notin (\lambda_1, \lambda_2)$ there is also at most one eigenvalue μ of B. In both cases $\dim \mathfrak{L}_\mu(B) = \dim \ker(B - \mu) = 1$. If $0 \in (\lambda_1, \lambda_2)$ there are at most two eigenvalues of B in (λ_1, λ_2). For every such eigenvalue $\mu \in (\lambda_1, \lambda_2)$ of B we have $\dim \mathfrak{L}_\mu(B) = \dim \ker(B - \mu) = 1$.*

Proof. (i) and (ii) are consequences of Theorem 5.12(ii) and Theorem 5.18(ii) and (iii). As B is non-negative, (iii) is a consequence of Theorem 3.1. (iv) follows from Theorem 4.7.

We prove (v). Let $0 \in \sigma_p(A)$ and let $\lambda_1 < \lambda_2$ be two consecutive eigenvalues of A in I. Then $n_A((\lambda_1, \lambda_2)) = n_{A,B}((\lambda_1, \lambda_2)) = 0$ and hence $n_B((\lambda_1, \lambda_2)) \leq 1$ by Theorem 5.12(i) (note that $0 \notin (\lambda_1, \lambda_2)$). The same holds in the case that $0 \in \rho(A)$ and $0 \notin (\lambda_1, \lambda_2)$. Moreover, we see $\dim \ker(B - \mu) = 1$ by Theorem 3.1 and $\mu \in \rho(A)$. Since B is non-negative, it has no Jordan chain of length ≥ 2 at non-zero points, cf. Remark 1.15. Finally, let $0 \in \rho(A)$ and $0 \in (\lambda_1, \lambda_2)$. Then $\lambda_1 < 0 < \lambda_2$ and $(\lambda_1, \lambda_2) \subseteq \rho(A)$. Hence, $n_B((\lambda_1, \lambda_2)) \leq 2$ by Theorem 5.12(ii) since $n_A(I) = n_{A,B}(I) = 0$. The remaining statements follow again from Theorem 3.1 as B is non-negative. □

Theorem 6.4. *Let A, B, and I be as in Assumption* I. *Assume in addition that $m_A(I) < \infty$ and*

$$A \text{ is non-negative}, \quad B \text{ has one negative square}, \quad \text{and} \quad 0 \notin I.$$

Then the following hold.

(i) *For the number of distinct eigenvalues of B in I we have*

$$n_A(I) - n_{A,B}(I) - 1 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + 3.$$

(ii) *For the total multiplicity of eigenvalues of B in I we have*

$$m_A(I) - 1 \leq m_B(I) \leq m_A(I) + 3.$$

(iii) *For every $\mu \in I$ the dimension of $\ker(B-\mu)$ is either one less, equal to, or one greater than the dimension of $\ker(A-\mu)$. If the dimension of $\ker(B-\mu)$ is greater than the dimension of $\ker(A-\mu)$ then B may have one Jordan chain of length 2 or 3 at μ. In the other cases B has no Jordan chain of length ≥ 2 at μ. In particular, we have*

$$|\dim \mathfrak{L}_\mu(A) - \dim \mathfrak{L}_\mu(B)| \leq 3.$$

(iv) *If $\sigma(B)$ is not contained in $\mathbb{R}$ then there exists $\eta \in \mathbb{C}^+$ such that*

$$\sigma(B) = (\sigma(B) \cap \mathbb{R}) \cup \{\eta, \overline{\eta}\}$$

and $\dim \mathfrak{L}_\eta(B) = \dim \mathfrak{L}_{\overline{\eta}}(B) = 1$. In this case we have for every $\mu \in I$ that the dimension of $\ker(B-\mu)$ is either one less, equal to, or one greater than the dimension of $\ker(A-\mu)$ and B has no Jordan chain of length ≥ 2 at μ. Between two consecutive eigenvalues of A in I there is at most one eigenvalue μ of B and $\dim \mathfrak{L}_\mu(B) = \dim\ker(B-\mu) = 1$.

(v) *If $\sigma(B) \subseteq \mathbb{R}$ then there exists at most one pair of consecutive eigenvalues $\lambda_1 < \lambda_2$ of A in I such that $m_B((\lambda_1,\lambda_2)) \geq 2$. For every eigenvalue $\mu \in (\lambda_1,\lambda_2)$ of B we have $\dim\ker(B-\mu) = 1$ and B has at most one Jordan chain of length ≥ 2 in (λ_1,λ_2) which then is of length ≤ 3. Between all remaining consecutive eigenvalues of A in I there is at most one eigenvalue μ of B and $\dim \mathfrak{L}_\mu(B) = \dim\ker(B-\mu) = 1$.*

Proof. (i) and (ii) hold by Theorems 5.12(ii) and 5.18(ii). (iii) follows from Theorem 3.1, Lemma 4.11, Corollary 4.10, and Theorem 4.12. In the following we assume $I \subseteq (0,\infty)$; the case $I \subseteq (-\infty,0)$ follows analogously.

We show (iv). The first assertion follows from Remark 4.8. For $\mu \in I$ we have by Corollary 1.14(ii) that $\mu \in \sigma_{++}(B)$ and hence B has no Jordan chain of length ≥ 2 at μ. Then $|\dim \mathfrak{L}_\mu(A) - \dim \mathfrak{L}_\mu(B)| \leq 1$ holds by Theorem 3.1. Now let $\lambda_1 < \lambda_2$ be two consecutive eigenvalues of A and assume there exist eigenvalues $\mu_1, \mu_2 \in (\lambda_1,\lambda_2) \cap \sigma(B)$, $\mu_1 < \mu_2$. Then $\mu_1, \mu_2 \in \sigma_{++}(B)$ by Corollary 1.14(ii) and by Proposition 2.15 there exists an eigenvalue λ_0 of A in $(\mu_1,\mu_2) \subseteq (\lambda_1,\lambda_2) \subseteq \rho(A)$, a contradiction. If μ is an eigenvalue of B between two consecutive eigenvalues of A then $\dim \mathfrak{L}_\mu(B) = 1$ follows as $\mu \in \rho(A) \cap \sigma_{++}(B)$.

(v) Let $m_B((\lambda_1,\lambda_2)) \geq 2$ for two consecutive eigenvalues $\lambda_1 < \lambda_2$ of A. If there exists an eigenvalue $\mu \in (\lambda_1,\lambda_2)$ of B with $\dim \mathfrak{L}_\mu(B) \geq 2$ then B has a Jordan chain of length ≥ 2 at μ and hence μ has a neutral eigenvector. Note that by Theorem 1.13(ii) this Jordan chain is of length at most 3. With Lemma 2.13(iii) we see $M_A'(\mu) = 0$. If there exists no such $\mu \in (\lambda_1,\lambda_2)$, then B has two (or more) distinct eigenvalues μ_1, μ_2 in (λ_1,λ_2). As M_A is holomorphic in (λ_1,λ_2) we see that $M_A'(\mu_1) \leq 0$ or $M_A'(\mu_2) \leq 0$. Hence, if $m_B((\lambda_1,\lambda_2)) \geq 2$ there exists a positive zero of M_A with non-positive derivative. By Proposition 5.10(ii) there exists only one such exceptional zero which shows that there exists at most one pair of consecutive eigenvalues of A with $m_B((\lambda_1,\lambda_2)) \geq 2$ and between all remaining pairs of consecutive eigenvalues there is at most one eigenvalue of B with $\dim\ker(B-\mu) = 1$ and there is no Jordan chain of length ≥ 2 at μ. $\square$

Theorem 6.5. *Let A, B, and I be as in Assumption* I. *Assume in addition that $m_A(I) < \infty$ and*

$$A \text{ is non-negative}, \quad B \text{ has one negative square}, \quad \text{and} \quad 0 \in I.$$

Then the following hold.

(i) *For the number of distinct eigenvalues of B in I we have*

$$n_A(I) - n_{A,B}(I) - 2 \leq n_B(I) \leq n_A(I) + n_{A,B}(I) + 3.$$

(ii) *For the total multiplicity of eigenvalues of B in I we have*

$$\begin{aligned} m_A(I) - 2 \leq m_B(I) \leq m_A(I) + 3 \quad & \text{if } 0 \notin \sigma_p(A), \\ m_A(I) - 4 \leq m_B(I) \leq m_A(I) + 6 \quad & \text{if } 0 \in \sigma_p(A). \end{aligned}$$

(iii) *For every $\mu \in I\backslash\{0\}$ the dimension of $\ker(B-\mu)$ is either one less, equal to, or one greater than the dimension of $\ker(A-\mu)$. If the dimension of $\ker(B-\mu)$ is greater than the dimension of $\ker(A-\mu)$ then B may have one Jordan chain of length 2 or 3 at μ. In the other cases B has no Jordan chain of length ≥ 2 at μ. In particular, we have*

$$|\dim \mathfrak{L}_\mu(A) - \dim \mathfrak{L}_\mu(B)| \leq 3.$$

(iv) *The Jordan structure of B at 0, i.e. the number and length of all Jordan chains of B at 0 compared with the Jordan chains of A at 0, is described in Proposition 6.1. In particular, there are 11 possible Jordan structures of B at 0, which are given in Table 4.1.*

(v) *If $\sigma(B)$ is not contained in $\mathbb{R}$ then there exists $\eta \in \mathbb{C}^+$ such that*

$$\sigma(B) = (\sigma(B) \cap \mathbb{R}) \cup \{\eta, \overline{\eta}\}$$

and $\dim \mathfrak{L}_\eta(B) = \dim \mathfrak{L}_{\overline{\eta}}(B) = 1$. In this case we have:

(a) *B has no Jordan chain of length ≥ 2 at $\mu \in I\backslash\{0\}$.*

(b) *The Jordan structure of B at 0, i.e. the number and length of all Jordan chains of B at 0 compared with the Jordan chains of A at 0 is described in Proposition 6.1. In this case the number of possible Jordan structures of B at 0 reduces to 7, namely Cases $1, 4, 7, 13, 18, 19,$ and 20 in Table 4.1.*

(c) *If $0 \in \sigma_p(A)$ then between two consecutive eigenvalues of A in I there is at most one eigenvalue μ of B and $\dim \mathfrak{L}_\mu(B) = \dim\ker(B-\mu) = 1$.*

(d) *If $0 \in \rho(A)$ then between two consecutive eigenvalues $\lambda_1 < \lambda_2$ of A in I such that $0 \notin (\lambda_1, \lambda_2)$ there is at most one eigenvalue μ of B and $\dim \mathfrak{L}_\mu(B) = \dim\ker(B-\mu) = 1$. If $0 \in (\lambda_1, \lambda_2)$ there are at most two eigenvalues of B in (λ_1, λ_2). For* *every such eigenvalue* $\mu \in (\lambda_1, \lambda_2)$ *of* B *we have* $\dim \mathfrak{L}_\mu(B) = \dim \ker(B - \mu) = 1$ *and Jordan chains of* B *at* 0 *are of length at most* 2.

(vi) *If* $\sigma(B) \subseteq \mathbb{R}$ *the following hold.*

(*a*) *If* $0 \in \sigma_p(A)$ *then there is at most one pair of consecutive eigenvalues* $\lambda_1 < \lambda_2$ *of* A *in* I *such that* $m_B((\lambda_1, \lambda_2)) \geq 2$. *For every eigenvalue* $\mu \in (\lambda_1, \lambda_2)$ *of* B *we have* $\dim \ker(B - \mu) = 1$ *and* B *has at most one Jordan chain of length* ≥ 2 *in* (λ_1, λ_2) *which then is of length* ≤ 3. *Between all remaining consecutive eigenvalues of* A *in* I *there is at most one eigenvalue* μ *of* B *and* $\dim \mathfrak{L}_\mu(B) = 1$.

(*b*) *If* $0 \in \rho(A)$ *denote by* $\lambda_+(\lambda_-)$ *the minimal positive* (*maximal negative, respectively*) *eigenvalue of* A. *Then there exists at most one pair of consecutive eigenvalues* $\lambda_1 < \lambda_2$ *of* A *in* $I \backslash (\lambda_-, \lambda_+)$ *such that* $m_B((\lambda_1, \lambda_2)) \geq 2$. *If such a pair exists then* B *has no eigenvalues in* (λ_-, λ_+), *for every eigenvalue* $\mu \in (\lambda_1, \lambda_2)$ *of* B *we have* $\dim \ker(B - \mu) = 1$, *and* B *has at most one Jordan chain of length* ≥ 2 *in* (λ_1, λ_2) *which then is of length* ≤ 3. *If no such pair* λ_1, λ_2 *exists then* $m_B((\lambda_-, \lambda_+)) \leq 2$. *In both cases, there is at most one eigenvalue* μ *of* B *between each remaining pair of consecutive eigenvalues of* A *in* $I \backslash (\lambda_-, \lambda_+)$ *and* $\dim \mathfrak{L}_\mu(B) = \dim \ker(B - \mu) = 1$.

Proof. (i) and (ii) follow from Theorem 5.12(ii) and Theorem 5.18(ii) and (iii). Statement (iii) follows from Theorems 1.13(ii) and 3.1, Lemma 4.11, and Corollary 4.10. (iv) follows from Theorem 4.7.

We show (v). The existence of η with the desired properties follows from Remark 4.8 and according to Remark 1.16 $B \upharpoonright (\mathfrak{L}_\mu [\dot{+}] \mathfrak{L}_{\overline{\mu}})^{[\perp]}$ is non-negative. Hence, the Jordan chains of B are of length one at real non-zero points in the spectrum and of length ≤ 2 at 0 by Corollary 1.14 and Remark 1.15. Together with Theorem 3.1 this shows (a) and (b).

Let $\lambda_1 < \lambda_2$ be two consecutive eigenvalues of A and let $0 \in \sigma_p(A)$. Assume there exist eigenvalues $\mu_1, \mu_2 \in (\lambda_1, \lambda_2) \cap \sigma(B)$, $\mu_1 < \mu_2$. Then $\mu_1, \mu_2 \in \sigma_{++}(B)$ by Corollary 1.14(ii) and by Proposition 2.15 there exists an eigenvalue λ_0 of A in $(\mu_1, \mu_2) \subseteq (\lambda_1, \lambda_2) \subseteq \rho(A)$, a contradiction. This shows (c). The same reasoning holds if $0 \in \rho(A)$ and $0 \notin (\lambda_1, \lambda_2)$. Hence, let $0 \in \rho(A) \cap (\lambda_1, \lambda_2)$. Then $\lambda_1 < 0 < \lambda_2$, $(\lambda_1, \lambda_2) \subseteq \rho(A)$, and we have $(\lambda_1, 0) \cap \sigma(B) \subseteq \sigma_{--}(B)$ and $(0, \lambda_2) \cap \sigma(B) \subseteq \sigma_{++}(B)$. Assume that there exist consecutive eigenvalues $\mu_1 < \mu_2 < \mu_3$ of B in (λ_1, λ_2). With Lemma 2.13 we see that μ_i is a zero of M_A with $M_A'(\mu_i) > 0$ ($M_A'(\mu_i) < 0$) if $\mu_i > 0$ ($\mu_i < 0$, respectively), $i = 1, 2, 3$. Since $M_A \in \mathcal{D}_0$ is holomorphic in (λ_1, λ_2) this is only possible if $\mu_1 < 0$, $\mu_2 = 0$, $\mu_3 > 0$, and $M_A''(0) < 0$ in contradiction to Lemma 5.11(iii). The remaining statements follow with (a) and (b).

(vi) The statements in (a) follow as in Theorem 6.4(v) as M_A has only one exceptional zero in $I \backslash \{0\}$. (b) follows from Corollary 5.20, Theorem 3.1 and Theorem 1.13(ii). □

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